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Enumerating minimal dominating sets in chordal graphs

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ABSTRACT

The maximum number of minimal dominating sets in a chordal graph on n vertices is known to be at most 1.6181^n. However, no example of a chordal graph with more than 1.4422^n minimal dominating sets is known. In this paper, we narrow this gap between the known upper and lower bounds by showing that the maximum number of minimal dominating sets in a chordal graph is at most 1.5214^n.

1. Introduction

Enumerating the minimal dominating sets of a graph is one of the most studied and important problems within enumeration algorithms, especially since it corresponds to the problem of enumerating the minimal transversals of a hypergraph [8]. Whether this enumeration can be performed in output polynomial time is one of the biggest and longest standing open problems within enumeration algorithms. Many special cases have been studied throughout the last decades, and recently a polynomial delay algorithm for chordal graphs has been given by Kanté et al. [9]. The problem has also been handled with exponential-time branching algorithms, which at the same time give an upper bound on the maximum number of minimal dominating sets a graph can have. On arbitrary graphs with n vertices, the upper bound is shown by Fomin et al. [5] to be 1.7159^n, whereas no example of a graph that has more than 1.5704^n minimal dominating sets is known [5]. This gap between the known upper and lower bounds has triggered a flow of research on special cases, and on several graph classes one has found the exact bound for the maximum number of minimal dominating sets [2,3]. On chordal graphs, however, a gap has remained, with the best known upper and lower bounds being respectively 1.6181^n and 1.4422^n, shown by Couturier et al. [2].

In this paper, we improve the upper bound, and we show that a chordal graph has at most 1.5214^n minimal dominating sets. An improved upper bound is of particular interest in light of the above mentioned recent polynomial delay algorithm for enumerating the minimal dominating sets of chordal graphs. It shows that such a polynomial delay algorithm will never spend more than $O(1.5214^n)$ time. It should be noted that chordal graphs form one of the most famous and widely studied graph classes, with a large variety of application areas [1,7].

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2. Preliminaries

We work with undirected and simple graphs. Such a graph $G = (V, E)$ is *non-trivial* if it contains at least one edge. The set of neighbors of a vertex $v \in V$ is denoted by $N(v)$, and $N[v] = N(v) \cup \{v\}$. The subgraph of G induced by a set of vertices $U \subseteq V$ is denoted by $G[U]$, and we use $G - v$ to denote $G[V \setminus \{v\}]$. A *clique* in G is a set of vertices that are all pairwise adjacent. A vertex v is *simplicial* if $N(v)$ is a clique.

A graph is *chordal* if it contains no induced (chordless) cycle of length 4 or more as an induced subgraph. Observe that induced subgraphs of chordal graphs are also chordal. A chordal graph that is not complete contains at least two non-adjacent simplicial vertices [4]. If two simplicial vertices u, v of G are adjacent, then it is easy to see that $N[u] = N[v]$; in this case we call u and v *simplicial twins*. We call a non-simplicial vertex v of a chordal graph G *almost-simplicial* if v has exactly one simplicial neighbor u, such that v is simplicial in $G - u$.

Observation 1. *Let G be a chordal graph and let u be a simplicial vertex of degree at least 2. If v is an almost-simplicial neighbor of u, then $N[v] \subseteq N[w]$ for every vertex $w \in N(u) \cap N(v)$.*

Simplicial vertices in chordal graphs could be arbitrarily far from each other; a simple path is an example. We show that, under some conditions, we can always find pairs of simplicial vertices that are close to each other. Our enumeration algorithm is heavily based on the following theorem, which could be of interest by itself.

Theorem 2. *If G is a non-trivial chordal graph, then there exists a pair of vertices u and v satisfying at least one of the following:*

1. *u and v are both simplicial vertices, and $N(u) \cap N(v) \neq \emptyset$;*
2. *u and v are simplicial twins;*
3. *u is simplicial, v is an almost-simplicial neighbor of u, and the degree of v in G is at least 2.*

Proof. If one of the first two conditions is satisfied, we are done. Assume that neither of the first two condition is satisfied, i.e., every pair of simplicial vertices have disjoint neighborhoods, and no two simplicial vertices are adjacent. Remove all simplicial vertices of G, which results in a non-empty chordal graph G'. Let v be a simplicial vertex of G'. Since the first condition was not satisfied, v was adjacent in G to at most one simplicial vertex. Since v was not simplicial in G, it was almost-simplicial, and hence it had exactly one simplicial neighbor u. Since the second condition was not satisfied, v cannot be of degree 0 in G', since otherwise it would be the simplicial twin of u in G. Thus the degree of v in G was at least 2. Consequently, u and v satisfy the third condition.

The following corollary of Theorem 2 will also be useful.

Corollary 3. *Let G be a non-trivial chordal graph without simplicial twins, in which every simplicial vertex is of degree 1, and no two vertices of degree 1 have a common neighbor. Then at least one of the following holds:*

1. *There exist two simplicial vertices u and u' with (distinct) almost-simplicial neighbors v and v', respectively, such that $N[v] \cap N[v'] \neq \emptyset$.*
2. *There exist three vertices u, v, and w, such that u is simplicial in G with an almost-simplicial neighbor v, and $w \in N(v)$ is simplicial in $G - v$ and has degree at least 2 in $G - u$.*

Proof. Since G is non-trivial and has no simplicial twins, G is not complete. Thus G has at least two non-adjacent simplicial vertices, say u and u'. By the premises of the corollary, $N(u) \cap N(u') = \emptyset$. Since every simplicial vertex has at least one non-simplicial neighbor, removing all simplicial vertices yields a non-trivial chordal graph G'. Furthermore, every simplicial vertex of G' was an almost-simplicial vertex in G. Consequently, applying Theorem 2 to G' completes the proof.

A vertex v is said to *dominate* the vertices in $N[v]$. A vertex set $S \subseteq V$ is a *dominating set* of G if $N[S] = V$. A dominating set S is *minimal* if no proper subset of S is a dominating set. If S is a minimal dominating set, then for every vertex $v \in S$, there is a vertex $u \in N[v]$ which is dominated only by v. We call such a vertex u a *private neighbor* of v; note that a vertex in S might be its own private neighbor.

In our algorithm, we will use the following observation by Couturier et al. [2] and some basic elements of their algorithm for enumerating the minimal dominating sets of chordal graphs. However, all our major branching rules are new, and they are based on Theorem 2.

Observation 4 *([2]). A simplicial vertex u cannot belong to a minimal dominating set containing a neighbor of u.*

3. Enumeration algorithm

For enumerating the minimal dominating sets of an input chordal graph $G = (V, E)$, we describe a recursive branching algorithm called $\textsc{mds}(U, D)$, with $U \subseteq V$ and $D \subseteq V \setminus U$. The algorithm generates all minimal dominating sets S of G satisfying $D \subseteq S$ and $S \cap (V \setminus U) = D$. The initial call $\textsc{mds}(V, \emptyset)$ will thus ensure that all minimal dominating sets of G are enumerated. By the description of the algorithm it will follow that every vertex of $V \setminus U$ is either dominated by D or guaranteed to be dominated in every following branch of the algorithm. Hence the vertices in $V \setminus U$ are not relevant for the further choices to be made by the algorithm. At every step of the algorithm we pick a simplicial vertex of $G[U]$ and branch on possibilities around this vertex. A vertex of $G[U]$ is said to be *dominated* if it is adjacent in G to a vertex of D.

The following branching rules are applied successively in the sense that a branching rule is applied only when all the conditions assumed in previous branching rules do not hold. When we speak about neighborhoods and degrees, we always refer to $G[U]$ unless otherwise stated. At any