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Decidability of order-based modal logics ☆

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ABSTRACT

Decidability of the validity problem is established for a family of many-valued modal logics, notably Gödel modal logics, where propositional connectives are evaluated according to the order of values in a complete sublattice of the real unit interval $[0, 1]$, and box and diamond modalities are evaluated as infima and suprema over (many-valued) Kripke frames. If the sublattice is infinite and the language is sufficiently expressive, then the standard semantics for such a logic lacks the finite model property. It is shown here, however, that, given certain regularity conditions, the finite model property holds for a new semantics for the logic, providing a basis for establishing decidability and PSPACE-completeness. Similar results are also established for S5 logics that coincide with one-variable fragments of first-order many-valued logics. In particular, a first proof is given of the decidability and co-NP-completeness of validity in the one-variable fragment of first-order Gödel logic.

1. Introduction

Many-valued modal logics extend the Kripke frame setting of classical modal logic with a many-valued semantics at each world and a many-valued or crisp (Boolean-valued) accessibility relation to model modal notions such as necessity, belief, and spatio-temporal relations in the presence of uncertainty, possibility, or vagueness. Applications include modelling fuzzy belief [17,22], spatial reasoning with vague predicates [33], many-valued tense logics [12], and fuzzy similarity measures [18]. Fuzzy description logics may also be interpreted, analogously to the classical case, as many-valued multi-modal logics (see, e.g., [5,21,35]).

Quite general approaches to many-valued modal logics, focussing largely on decidability and axiomatization issues for finite-valued modal logics, are described in [6,15,16,30]. For modal logics based on an infinite-valued semantics, typically over the real unit interval $[0, 1]$, two core families can be identified. Many-valued modal logics of "magnitude" are based on a semantics related to Łukasiewicz infinite-valued logic with connectives interpreted by continuous functions over real numbers [13,19,23]. Typical many-valued modal logics of the second family are based instead on the semantics of infinite-

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valued Gödel logics [9,10,19,27]. The standard infinite-valued Gödel logic (also known as Gödel–Dummett logic) interprets truth values as elements of $[0,1]$, conjunction and disjunction as minimum and maximum, respectively, and implication $x \to y$ as y for $x > y$ and 1 otherwise. Modal operators $\Box$ and $\Diamond$ (not inter-definable in this setting) are interpreted as infima and suprema of values at accessible worlds. More generally, "order-based" modal logics may be defined over a complete sublattice of $[0,1]$ with additional operations depending only on the order.

Propositional Gödel logic has been studied intensively both as a fundamental "t-norm based" fuzzy logic [19,28] and as an intermediate (or superintuitionistic) logic, obtained as an extension of an axiomatization of propositional intuitionistic logic with the prelinearity axiom schema $(\varphi \to \psi) \vee (\psi \to \varphi)$. The many-valued modal logics considered in this paper diverge considerably, however, from the modal intermediate logics investigated in [36] (and elsewhere), which use two accessibility relations for Kripke models, one for the modal operators and one for the intuitionistic connectives. We remark also that, unlike the operations added to infinite-valued logics in [11,20], which represent truth stressers such as "very true" or "classically true", the modalities considered here cannot be interpreted simply as unary connectives on the real unit interval $[0,1]$.

The first main contribution of this paper is to establish PSPACE-completeness results (matching the complexity of the classical modal logic K [25]) for the validity problem of Gödel modal logics and other order-based modal logics defined over complete sublattices of $[0,1]$ satisfying certain local regularity conditions (e.g., sublattices order-isomorphic to the positive integers with an added top element and the negative integers with an added bottom element). The finite model property typically fails even for the box and diamond fragments of these logics. Decidability and PSPACE-completeness of the validity problem for these fragments of Gödel modal logics over $[0,1]$ was established in [27] using analytic Gentzen-style proof systems, but this methodology does not seem to extend easily to the full logics. Here, alternative Kripke semantics are provided for order-based modal logics that not only have the same valid formulas as the original semantics, but also admit the finite model property. The key idea of this new semantics is to restrict evaluations of modal formulas at a world to a particular set of truth values.

The second main contribution of the paper is to establish co-NP-completeness results for the validity problem of crisp order-based "S5" logics: order-based modal logics where accessibility is an equivalence relation. Such logics may be interpreted also as one-variable fragments of first-order many-valued logics. In particular, the open decidability problem for validity in the one-variable fragment of first-order Gödel logic (see, e.g., [19, Chapter 9, Problem 13]) is answered positively and shown to be co-NP-complete. This result matches the complexity of the one-variable fragments of classical first-order logic (equivalently, S5) and first-order Łukasiewicz logic (see [19]), and contrasts with the co-NEXPTIME-completeness of the one-variable fragment of first-order intuitionistic logic (equivalently, the intuitionistic modal logic MIPC) [26].

2. Order-based modal logics

We consider "order-based" modal logics where propositional connectives are interpreted at individual worlds in an algebra consisting of a complete sublattice of $\langle [0,1], \wedge, \vee, 0, 1\rangle$ with operations defined based only on the order. Modalities $\Box$ and $\Diamond$ are defined using infima and suprema, respectively, according to either a (crisp, i.e., Boolean-valued) binary relation on the set of worlds or a binary mapping (many-valued relation) from worlds to values of the algebra. For convenience, we consider only finite algebraic languages, noting that to decide the validity of a formula we may in any case restrict to the language containing only operation symbols occurring in that formula.

We reserve the symbols $\Rightarrow$, &, $\sim$, and $\approx$ to denote implication, conjunction, negation, and equality, respectively, in classical first-order logic. We also recall an appropriate notion of first-order definability of operations for algebraic structures. Let $\mathcal{L}$ be an algebraic language, $\mathbf{A}$ an algebra for $\mathcal{L}$, and $\mathcal{L}'$ a sublanguage of $\mathcal{L}$. An operation $f\colon A^n \to A$ is *defined* in $\mathbf{A}$ by a first-order $\mathcal{L}'$-formula $F(x_1,\dots,x_n,y)$ with free variables $x_1,\dots,x_n,y$ if for all $a_1,\dots,a_n,b \in A$,

$$\mathbf{A} \models F(a_1,\dots,a_n,b) \quad \Leftrightarrow \quad f(a_1,\dots,a_n) = b.$$

2.1. Order-based algebras

Let $\mathcal{L}$ be a finite algebraic language that includes the binary operation symbols $\wedge$ and $\vee$ and constant symbols $\bar{0}$ and $\bar{1}$ (to be interpreted by the usual lattice operations), and denote the finite set of constants (nullary operation symbols) of this language by $\mathrm{C}_{\mathcal{L}}$. An algebra $\mathbf{A}$ for $\mathcal{L}$ will be called *order-based* if it satisfies the following conditions:

(1) $\langle A, \wedge^{\mathbf{A}}, \vee^{\mathbf{A}}, 0, 1\rangle$ is a complete sublattice of $\langle [0,1], \min, \max, 0, 1\rangle$; i.e., $\{0,1\} \subseteq A \subseteq [0,1]$ and for all $B \subseteq A$, $\bigwedge^{[0,1]} B$ and $\bigvee^{[0,1]} B$ belong to A.
(2) For each operation symbol $\star$ of $\mathcal{L}$, the operation $\star^{\mathbf{A}}$ is definable in $\mathbf{A}$ by a quantifier-free first-order formula in the algebraic language consisting of $\wedge$, $\vee$, and constants from $\mathrm{C}_{\mathcal{L}}$.

We also let $\mathrm{C}^{\mathbf{A}}_{\mathcal{L}}$ denote the finite set of constant operations $\{c^{\mathbf{A}} : c \in \mathrm{C}_{\mathcal{L}}\}$ and define $R(\mathbf{A})$ and $L(\mathbf{A})$ to be the sets of *right* and *left accumulation points*, respectively, of $\mathbf{A}$ in the usual topology inherited from $[0,1]$; that is,