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Boolean dependence logic and partially-ordered connectives

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A B S T R A C T

We introduce a new variant of dependence logic ($\mathcal{D}$) called Boolean dependence logic ($\mathcal{BD}$). In $\mathcal{BD}$ dependence atoms are of the type $=(x_1, \ldots, x_n, \alpha)$, where α is a Boolean variable. Intuitively, with Boolean dependence atoms one can express quantification of relations, while standard dependence atoms express quantification over functions. We compare the expressive powers of $\mathcal{BD}$ to $\mathcal{D}$ and first-order logic enriched by partially-ordered connectives, $\mathcal{FO}(\mathcal{POC})$. We show that the expressive power of $\mathcal{BD}$ and $\mathcal{D}$ coincide. We define natural syntactic fragments of $\mathcal{BD}$ and show that they coincide with the corresponding fragments of $\mathcal{FO}(\mathcal{POC})$ with respect to expressive power. We then show that the fragments form a strict hierarchy. We also gain a new characterization for $\mathcal{SNP}$.

1. Introduction

Dependence is an important concept in various scientific disciplines. A multitude of formalisms have been designed to model dependences, for example, in database theory, social choice theory, and quantum mechanics. However, for a long time the research has been scattered and the same ideas have been discovered many times over in different fields of science. One important reason, albeit surely not the only one, for this scatteredness was the lack of a unified logical background theory for the concept of dependence. Over the last decade the emergence of dependence logic and the extensive and rigorous research conducted on dependence logic and related formalisms have mended this shortcoming.

Dependences between variables in formulae is the most direct way to model dependences in logical systems. In first-order logic the order in which quantifiers are written determines dependence relations between variables. For example, when using game theoretic semantics to evaluate the formula

$$\forall x_0 \exists x_1 \forall x_2 \exists x_3\, \varphi,$$

the choice for x_1 depends on the value for x_0, and the choice for x_3 depends on the value of both universally quantified variables x_0 and x_2. The first to consider more complex dependences between variables was Henkin [22] with his partially-ordered quantifiers. The simplest non-trivial partially-ordered quantifier is usually written in the form

$$\begin{pmatrix} \forall x_0 & \exists x_1 \\ \forall x_2 & \exists x_3 \end{pmatrix} \varphi, \qquad (1)$$

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and the idea is that x_1 depends only on x_0 and x_3 depends only on x_2. Enderton [13] and Walkoe [43] observed that exactly the properties definable in existential second-order logic ($\mathcal{ESO}$) can be expressed with partially-ordered quantifiers. Building on the ideas of Henkin, Blass and Gurevich introduced in [4] the narrow Henkin quantifiers

$$\begin{pmatrix} \forall \vec{x}_1 & \exists \alpha_1 \\ \vdots & \vdots \\ \forall \vec{x}_n & \exists \alpha_n \end{pmatrix} \varphi.$$

Here $\alpha_1, \ldots, \alpha_n$ are Boolean variables (or, more generally, variables ranging over some fixed finite domains). The idea of Blass and Gurevich was further developed by Sandu and Väänänen in [35] where they introduced partially-ordered connectives

$$\begin{pmatrix} \forall \vec{x}_1 & \bigvee_{b_1 \in \{0,1\}} \\ \vdots & \vdots \\ \forall \vec{x}_n & \bigvee_{b_n \in \{0,1\}} \end{pmatrix} \gamma.$$

Here γ is a tuple $(\gamma_{b_1 \ldots b_n})_{(b_1,\ldots,b_n)\in\{0,1\}^n}$ of formulae, and the choice of each bit b_i determining the disjunct $\gamma_{b_1 \ldots b_n}$ to be satisfied depends only on $\vec{x}_i$.

The first to linearize the idea behind the syntax of partially-ordered quantifiers were Hintikka and Sandu [23,24], who introduced independence-friendly logic ($\mathcal{IF}$). $\mathcal{IF}$-logic extends $\mathcal{FO}$ in terms of so-called slashed quantifiers. Dependence logic ($\mathcal{D}$), introduced by Väänänen [40], was inspired by $\mathcal{IF}$-logic, but the approach of Väänänen provided a fresh perspective on quantifier dependence. In dependence logic the dependence relations between variables are written in terms of novel atomic dependence formulae. For example, the partially-ordered quantifier (1) can be expressed in dependence logic as follows:

$$\forall x_0 \exists x_1 \forall x_2 \exists x_3 (=(x_2, x_3) \wedge \varphi).$$

The atomic formula $=(x_2, x_3)$ has the explicit meaning that x_3 is completely determined by x_2 and nothing else.

Over the last decade the research related to independence-friendly logic and dependence logic has bloomed. A variety of closely related logics have been defined and various applications suggested, see e.g., [1,2,5,19,38,39]. Especially the research on propositional and modal dependence logics (see, e.g., [10,20,31,36,41]), and the study of fragments of first-order dependence logic (see, e.g., [7,11,27,28]) has been very active. For research on extensions of related logics, by fixpoint operators, see [6,18], and by generalized quantifiers or atoms see [14,15,29]. Furthermore, within the last six years eight PhD theses have been published on closely related topics, see [8,17,26,30,33,34,42,44]. See also the monographs [32,40]. Research related to partially-ordered connectives has been less active. For recent work, see e.g., [21,37].

In this article we introduce a new variant of dependence logic called Boolean dependence logic ($\mathcal{BD}$). Boolean dependence logic extends first-order logic with special restricted versions of dependence atoms which we call Boolean dependence atoms. While all variables occurring in dependence atoms

$$=(x_1, \ldots, x_n, y)$$

of dependence logic are first-order variables, in Boolean dependence atoms

$$=(x_1, \ldots, x_n, \alpha)$$

of Boolean dependence logic only the antecedents $x_1, \ldots, x_n$ are first-order variables, whereas the consequent α is a Boolean variable. A Boolean variable is a special kind of variable with values that range over the set $\{\top, \bot\}$, i.e., Boolean variables as assigned a value *true* or *false*.

Intuitively, the dependence atoms of dependence logic express quantification over functions. The meaning of the dependence atom

$$=(x_1, \ldots, x_n, y)$$

is that there exists an n-ary function that maps the values of the variables $x_1, \ldots, x_n$ to the value of the variable y. Analogously, Boolean dependence atoms can be interpreted as expressing quantification of relations or, more precisely, characteristic functions of relations. In this sense, the meaning of the Boolean dependence atom

$$=(x_1, \ldots, x_n, \alpha)$$

is that there exists a characteristic function of an n-ary relation that maps the values of the variables $x_1, \ldots, x_n$ to the value $\bot$ or $\top$. Since the expressive powers of dependence logic and existential second-order logic coincide, and since in existential second-order logic it is clear that functions and relations are interdefinable, the question arises whether there is any significant difference between dependence logic and Boolean dependence logic. In fact, in terms of expressive power there is no difference: in Section 6 we show that the expressive power of dependence logic and Boolean dependence logic coincide.

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