ARTICLE IN PRESS YJCSS:3043

Journal of Computer and System Sciences ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Computer and System Sciences

www.elsevier.com/locate/jcss

On the realisability of double-cross matrices by polylines in the plane

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A R T I C L E   I N F O

Article history:
Received 8 August 2015
Received in revised form 25 October 2016
Accepted 16 December 2016
Available online xxxx

Keywords:
Spatial reasoning
Double-cross calculus
Qualitative description of polylines
Computational algebraic geometry
Algorithmic complexity

A B S T R A C T

We study a decision problem, that emerges from the area of spatial reasoning. This decision problem concerns the description of polylines in the plane by means of their *double-cross matrix*. In such a matrix, the relative position of each pair of line segments in a polyline is expressed by means of a 4-tuple over $\{-, 0, +\}$. However, not any such matrix of 4-tuples is the double-cross matrix of a polyline. This gives rise to the decision problem: given a matrix of such 4-tuples, decide whether it is the double-cross matrix of a polyline. This problem is decidable, but it is NP-hard. In this paper, we give polynomial-time algorithms for the cases where consecutive line segments in a polyline make angles that are multiples of 90° or 45° and for the case where, apart from an input matrix, the successive angles of a polyline are also given as input.

1. Introduction and summary of results

Polylines arise in Geographical Information Science (GIS) in a multitude of ways. One recent example comes from the collection of moving object data, where trajectories of moving persons (or animals), that carry GPS-equipped devices, are collected in the form of time–space points that are registered at certain (ir)regular moments in time. The spatial trace of this movement is a collection of points in two-dimensional geographical space, that form a polyline, when in between the measured sample points, for instance, linear interpolation is applied [15]. Another example of the use of polylines comes from shape recognition and retrieval, which arises in domains, such as computer vision and image analysis. Here, closed polylines (or polygons) often occur as the boundary of two-dimensional shapes or regions.

In examples, such as the above, there are, roughly speaking, two very distinct approaches to deal with polygonal curves and shapes. On the one hand, there are the *quantitative* approaches and, on the other hand, there are the *qualitative* approaches. Initially, most research efforts have dealt with the quantitative methods [4,9,18,23]. Only afterwards, the qualitative approaches have gained more attention, mainly supported by research in cognitive science that provides evidence that qualitative models of shape representation are much more expressive than their quantitative counterpart and reflect better the way in which humans reason about their environment [12]. The principles behind qualitative approaches to deal with polylines are also related to the field of *spatial reasoning*, which has as one of its main objectives to present geographic information in a qualitative way, to facilitate reasoning about it. For an overview of spatial and for spatio-temporal reason-

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http://dx.doi.org/10.1016/j.jcss.2016.12.001

ing, we refer to Chapter 12 in [13]. The reason for using a *qualitative representation* is that the available information is often imprecise, partial and subjective [11].

One of the formalisms to qualitatively describe polylines in the plane is given by the *double-cross calculus*. In this method, a *double-cross matrix* captures the relative position (or orientation) of any two line segments in a polyline by describing it with respect to a double cross based on the starting points of these line segments [11,28]. For an overview of the use of constraint calculi in qualitative spatial reasoning, we refer to [24]. In the $N \times N$ double-cross matrix of a polyline with N line segments (or $N+1$ vertices), the relative position (or orientation) of two (oriented) line segments in a polyline is encoded by means of a 4-tuple, whose entries come from the set $\{-, 0, +\}$.

However, not every $N \times N$ matrix of 4-tuples from $\{-, 0, +\}$ is the double-cross matrix of a polyline with $N+1$ vertices. This gives rise to the following decision problem: *Given an $N \times N$ matrix of 4-tuples from $\{-, 0, +\}$, decide whether or not it is the double-cross matrix of a polyline (with $N+1$ vertices), and if it is, given an example of a polyline that realises the matrix.*

To start with, we give a known collection of polynomial (in)equalities on the coordinates of the vertices of a polyline, that express the information contained in the double-cross matrix of a polyline. Since first-order logic over the reals (or elementary geometry) is decidable [26], it follows that this decision problem is also decidable. However, we are left with the question of its time complexity.

In computational algebraic geometry, the problem can be viewed as a satisfiability problem of a system of quadratic equations in $2(N+1)$ variables. However, the known best algorithms to solve our problem (including the production of sample points) take exponential time. Our decision problem has many particularities: the polynomials are homogeneous of degree 2; they use few monomials and each of them uses only six variables. Nevertheless, the problem is known to be NP-hard [25,24]. Whether or not this problem is in NP is less obvious, since no a priori polynomial bound on the complexity of sample points (to be guessed) is obvious. We discuss this problem in more detail in Section 3.

In this paper, we focus on subclasses of the above decision problem, for which we can give *polynomial time* decision algorithms. A first subclass is obtained by restricting the attention to polylines in which consecutive line segments make angles that are multiples of 90°. For this sub-problem, we give a $O(N^2)$-time decision procedure. Next, we turn our attention to polylines in which consecutive line segments make angles that are multiples of 45°. To solve the more complicated case of 45°-polylines, we introduce the polar-coordinate representation of double-cross matrices. We give two-way translations between the Cartesian- and the polar-coordinate representations. Using polar coordinates, our decision problem can be reduced to a linear programming problem (with algebraic coefficients, however). For the particular decision problem of a double-cross matrix M being realisable (or not) by a 45°-polyline, we can make use of the fact that the entries of M above its diagonal give exact information on the angles that a polyline, that would realise M, should have. This one-to-one correspondence between the qualitative double-cross information and the angle information implies that our decision problem simplifies to deciding whether or not appropriate segment lengths of a polyline exist. The latter problem is a linear programming problem, that can be solved in polynomial time. In fact, we show that this situation can be generalised and we first show that whenever the consecutive angles of a polyline are given, it can be decided in polynomial time whether a matrix M can be realised by a polyline (with the given angle sequence). Next, we apply this more general result to the case of 45°-polylines to obtain a polynomial time decision procedure. This result has some implications on the convexity of the solution set consisting of all 45°-polylines that realise a matrix. It is not the intention of this paper to discuss implementations of and experiments with the proposed methods.

Organisation. This paper is organised as follows. Section 2 gives the definition of a polyline, the double-cross matrix of a polyline and the known results on the algebraic interpretation of the double-cross matrix. In Section 3, we state our decision problem in a more technical way and discuss some of its general properties. Section 4 gives a $O(N^2)$-time decision procedure for the case of 90°-polylines. In Section 5, we introduce the polar-coordinate representation of double-cross matrices. In Section 6, we use the double-cross conditions in polar form to show the existence of a polynomial-time realisability test in the case where, apart from an input matrix, also the successive angles of a polyline are given as input. Section 7 gives a polynomial-time decision procedure for the case of 45°-polylines. The paper ends with concluding remarks that include variants of our decision problem.

2. Definition and preliminaries

In this section, we give the definitions of a polyline, an α-polyline and of the double-cross matrix of a polyline. We also give an algebraic interpretation of the double-cross matrix.

We start with the following notational conventions. Let $\mathbf{R}$ denote the sets of the real numbers, and let $\mathbf{R}^2$ denote the two-dimensional real plane. To stress that some real values are constants, we use sans serif characters: $\mathsf{x}, \mathsf{y}, \mathsf{x}_0, \mathsf{y}_0, \mathsf{x}_1, \mathsf{y}_1, \ldots$. Real variables are denoted in normal characters. For constant points of $\mathbf{R}^2$, we use the sans serif characters $\mathsf{p}, \mathsf{p}_0, \mathsf{p}_1, \ldots$.

2.1. Polylines and α-polylines

The following definition specifies what we mean by *polylines*. We define a polyline as a finite sequence of points in $\mathbf{R}^2$ (which is often used as its finite representation). When we add the line segments between consecutive points we obtain what we call the *semantics* of the polyline. We also introduce some terminology about polylines.

Please cite this article in press as: B. Kuijpers, B. Moelans, On the realisability of double-cross matrices by polylines in the plane, J. Comput. Syst. Sci. (2016), http://dx.doi.org/10.1016/j.jcss.2016.12.001