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On groups generated by bi-reversible automata: The two-state case over a changing alphabet

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ABSTRACT

The notion of an automaton over a changing alphabet $X = (X_i)_{i\geq 1}$ is used to define and study automorphism groups of the tree X^* of finite words over X. The concept of bi-reversibility for Mealy-type automata is extended to automata over a changing alphabet. It is proved that a non-abelian free group can be generated by a two-state bi-reversible automaton over a changing alphabet $X = (X_i)_{i\geq 1}$ if and only if X is unbounded. The characterization of groups generated by a two-state bi-reversible automaton over the sequence of binary alphabets is established.

1. Transducers and groups generated by them

A changing alphabet is simply an infinite sequence $X = (X_i)_{i\geq 1}$ of non-empty finite sets X_i (so-called sets of letters or finite alphabets). If the sequence $(|X_i|)_{i\geq 1}$ of cardinalities of the finite alphabets is bounded, then X is called bounded, otherwise unbounded.

An automaton A (other names: transducer, time-varying automaton) over a changing alphabet $X = (X_i)_{i\geq 1}$ is a finite set Q (set of states) together with an infinite sequence $\varphi = (\varphi_i)_{i\geq 1}$ of transition functions $\varphi_i\colon Q \times X_i \to Q$ and an infinite sequence $\psi = (\psi_i)_{i\geq 1}$ of output functions $\psi_i\colon Q \times X_i \to X_i$. If all these sequences are constant, then A is called a Mealy-type automaton and it is usually identified with the quadruple $(X_1, Q, \varphi_1, \psi_1)$. If for each $i \geq 1$ and $q \in Q$ the mapping $\sigma_{i,q}\colon x \mapsto \psi_i(q,x)$ $(x \in X_i)$ is a permutation of the alphabet X_i, then A is called invertible. The mapping $\sigma_{i,q}$ $(i \geq 1,\ q \in Q)$ is called the labeling of a state q in the i-th transition of the automaton A.

It is convenient to interpret an automaton $A = (X, Q, \varphi, \psi)$ as a machine, which being at any moment $i \geq 1$ in a state $q \in Q$ and reading from the input tape a letter $x \in X_i$, types on the output tape the letter $\psi_i(q,x)$, goes to the state $\varphi_i(q,x)$ and moves both tapes to the next position. This interpretation naturally associates with each state $q \in Q$ the transformation $A_q\colon X^* \to X^*$ of the tree X^* of finite words over the changing alphabet $X = (X_i)_{i\geq 1}$. The tree X^* consists of finite sequences $x_1x_2\ldots x_t$ $(t \geq 1)$ such that $x_i \in X_i$ for every $1 \leq i \leq t$ (we also assume the empty word denoted by ϵ); if the sequence $(|X_i|)_{i\geq 1}$ is constant, then we obtain a so-called regular rooted tree. The mapping $A_q\colon X^* \to X^*$ is defined as follows: $A_q(\epsilon) := \epsilon$ and

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$$A_q(x_1x_2\dots x_t) := \psi_1(q_1, x_1)\psi_2(q_2, x_2)\dots\psi_t(q_t, x_t),$$

where the states $q_i \in Q$ are defined recursively $q_1 := q$ and $q_{i+1} := \varphi_i(q_i, x_i)$ for $1 \le i < t$.

If an automaton $A = (X, Q, \varphi, \psi)$ is invertible, then the inverse automaton $A^{-1} := (X, Q, \varphi', \psi')$ is defined as follows:

$$\varphi_i'(q, x) := \varphi_i(q, \sigma_{i,q}^{-1}(x)), \quad \psi_i'(q, x) := \sigma_{i,q}^{-1}(x)$$

for all $i \ge 1$, $q \in Q$ and $x \in X_i$. In this case the mappings A_q and A_q^{-1} ($q \in Q$) are mutually inverse automorphisms of the tree X^*; that is, they are mutually inverse permutations of the set X^* preserving the empty word and the vertex adjacency (we assume that two words are adjacent if one of them arises from the second by deleting the last letter). In particular, these permutations preserve the lengths and the common beginnings of words. We then refer to the group generated by the permutations A_q for $q \in Q$ (with the composition of mappings as a product) as the group generated by the automaton A and denote it by $G(A)$, i.e., we define

$$G(A) := \langle A_q : q \in Q \rangle.$$

Each group of the form $G(A)$ is finitely generated; and it is an example of a residually finite group (as the whole group $Aut(X^*)$ is residually finite – see [5]).

The potential for Mealy-type transducers was discovered in group theory about fifty years ago, when realized that quite simple formulae describing the transition and output functions in an automaton A may result in some exotic properties of the group $G(A)$. The flagship example is the famous Grigorchuk group generated by a five-state Mealy automaton over the binary alphabet (for more on interesting properties of groups generated by Mealy automata see the survey paper [5] or the monograph [11]). The notion of an automaton over a changing alphabet was introduced in [16], where we obtained a useful combinatorial tool to define and study automorphism groups of a spherically homogeneous rooted tree, which is not necessarily a regular rooted tree.

An important problem in the theory of groups generated by transducers is to verify which finitely generated abstract groups G can be realized as groups of the form $G(A)$, as well as to find simple and applicable formulae describing the corresponding automaton A. In particular, the problem of finding an explicit realization of a non-abelian free group turned out to be far from trivial. It was solved by Glasner and Mozes [3] in 2005, but even in 80s of the last century it was conjectured that the so-called Aleshin–Vorobets automaton generates a non-abelian free group of rank three, which M. Vorobets and Ya. Vorobets finally confirmed in [13].

The question on the existence of a 2-state automaton over a bounded changing alphabet which generates a non-abelian free group is still open. In particular, we do not know if there is a 2-state Mealy automaton generating $\mathcal{F}_2$ (non-abelian free group of rank two). Indeed, in all known realizations of non-abelian free groups by Mealy automata the generating automata have more than two states [3,11,12,14]. Apart from Mealy automata, there are two various realizations of $\mathcal{F}_2$ by a 2-state automaton over an unbounded changing alphabet (see Examples 1–2 in the next section). It turns out that all these transducers are bi-reversible.

2. Bi-reversible automata over a changing alphabet

The concept of reversibility and bi-reversibility for Mealy automata was introduced by Macedońska, Nekrashevych and Sushchanskyy [7], who found a connection between the group of automorphisms defined by bi-reversible automata over a finite alphabet X and the commensurator of the non-abelian free group $\mathcal{F}_{|X|}$. Further development was due to Glasner and Mozes [3], who associate with each bi-reversible Mealy automaton a square complex and its universal covering. This allowed to construct the first examples of Mealy automata generating non-abelian free groups.

Let us recall that a Mealy automaton $A = (X, Q, \varphi, \psi)$ is reversible if every letter $x \in X$ acts (via the transition function φ) like a permutation of the set of states, i.e. for each $x \in X$ the mapping $q \mapsto \varphi(q, x)$ ($q \in Q$) defines a permutation of the set Q. If A is invertible and both the automata A and A^{-1} are reversible, then A is called bi-reversible.

Recently, there has been a lot of interest in this kind of Mealy automata. For example, Bondarenko, D'Angeli and Rodaro [1] constructed a bi-reversible 3-state Mealy automaton over the ternary alphabet $X = \{0, 1, 2\}$ which generates the lamplighter group $\mathbb{Z}_3 \wr \mathbb{Z}$, providing the first example of a not finitely presented group generated by a bi-reversible Mealy automaton. In [6] Klimann dealt with the semigroups generated by 2-state reversible Mealy automata and showed that every such a semigroup is either finite or free. In [4] Godin and Klimann referred to the well-known Burnside problem and proved that connected reversible Mealy automata with a prime number of states can not generate infinite torsion groups. D'Angeli and Rodaro [2] associated with each bi-reversible Mealy automaton A an automaton $(\partial A)^-$, which they called the enriched dual of A, and next, they showed how the boundary dynamics of the semigroup generated by $(\partial A)^-$ characterizes the algebraic properties (in particular, the property of being not free) of the group $G(A)$.

It turns out that the notion of reversibility and bi-reversibility can be naturally extended to automata over a changing alphabet.

Definition 1. We call an automaton $A = (X, Q, \varphi, \psi)$ over a changing alphabet $X = (X_i)_{i \ge 1}$ reversible if for every $i \ge 1$ and every letter $x \in X_i$ the mapping $q \mapsto \varphi_i(q, x)$ ($q \in Q$) defines a permutation of the set Q. If A is invertible and both A and A^{-1} are reversible automata, then we call the automaton A bi-reversible.