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A multivariate framework for weighted FPT algorithms ☆

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ABSTRACT

We introduce a multivariate approach for solving weighted parameterized problems. By allowing flexible use of parameters, our approach defines a framework for applying the classic bounded search trees technique. In our model, given an instance of size n of a minimization/maximization problem, and a parameter $W \geq 1$, we seek a solution of weight at most/at least W. We demonstrate the usefulness of our approach in solving Vertex Cover, 3-Hitting Set, Edge Dominating Set and Max Internal Out-Branching. While the best known algorithms for these problems admit running times of the form $c^W n^{O(1)}$, for some $c > 1$, our framework yields running times of the form $c^s n^{O(1)}$, where $s \leq W$ is the minimum size of a solution of weight at most/at least W. If no such solution exists, $s = \min\{W, m\}$, where m is the maximum size of a solution. In addition, we analyze the parameter $t \leq s$, the minimum size of a solution.

1. Introduction

Many fundamental problems in graph theory are NP-hard already on *unweighted* graphs. This wide class includes, among others, Vertex Cover, 3-Hitting Set, Edge Dominating Set and Max Internal Out-Branching [1]. Fast existing parameterized algorithms for these problems, which often exploit the structural properties of the underlying graph, cannot be naturally extended to handle weighted instances. Thus, solving efficiently weighted graph problems has remained among the outstanding open questions in parameterized complexity, as excellently phrased by Hajiaghayi [2]:

> "Most fixed-parameter algorithms for parameterized problems are inherently about *unweighted* graphs. Of course, we could add integer weights to the problem, but this can lead to a huge increase in the parameter. Can we devise fixed-parameter algorithms for weighted graphs that have less severe dependence on weights? Is there a nice framework for designing fixed-parameter algorithms on weighted graphs?"

We answer these questions affirmatively, by developing a multivariate framework for solving weighted parameterized problems. We use this framework to obtain efficient algorithms for the following fundamental graph problems.

Weighted Vertex Cover (WVC): Given a graph $G = (V, E)$, a weight function $w : V \to \mathbb{Q}^{\geq 1}$, and a parameter $W \in \mathbb{Q}^{\geq 1}$, find a vertex cover $U \subseteq V$ (i.e., every edge in E has an endpoint in U) of weight at most W (if one exists).

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Weighted 3-Hitting Set (W3HS): Given a 3-uniform hypergraph $G = (V, E)$, a weight function $w : V \to \mathbb{Q}^{\geq 1}$, and a parameter $W \in \mathbb{Q}^{\geq 1}$, find a hitting set $U \subseteq V$ (i.e., every hyperedge in E has an endpoint in U) of weight at most W (if one exists).

Weighted Edge Dominating Set (WEDS): Given a graph $G = (V, E)$, a weight function $w : E \to \mathbb{Q}^{\geq 1}$, and a parameter $W \in \mathbb{Q}^{\geq 1}$, find an edge dominating set $U \subseteq E$ (i.e., every edge in E touches an endpoint of an edge in U) of weight at most W (if one exists).

Weighted Max Internal Out-Branching (WIOB): Given a directed graph $G = (V, E)$, a weight function $w : V \to \mathbb{Q}^{\geq 1}$, and a parameter $W \in \mathbb{Q}^{\geq 1}$, find an out-branching of G (i.e., a spanning tree having exactly one vertex of in-degree 0) having internal vertices of total weight at least W (if one exists).

Parameterized algorithms solve NP-hard problems by confining the combinatorial explosion to a parameter k. More precisely, a problem is *fixed-parameter tractable (FPT)* with respect to a parameter k if it can be solved in time $O^*(f(k))$ for some function f, where O^* hides factors polynomial in the input size n. In deriving our results, we rely on the standard assumption that W and element weights are at least 1 (indeed, this ensures fixed-parameter tractability with respect to W (see, e.g., [3])).

Existing FPT algorithms for the above problems have running times of the form $O^*(c^W)$. Using our framework, we obtain faster algorithms, whose running times are of the form $O^*(c^s)$, where $s \leq W$ is the minimum size of a solution of weight at most (at least) W. If no such solution exists, $s = \min\{W, m\}$, where m is the maximum size of a solution (for the unweighted version). We note that obtaining *slow* running times of this form is simple; the challenge lies in having the bases the same as in the previous best known $O^*(c^W)$ running times. Clearly, s can be significantly smaller than W. Moreover, almost all of the bases in our $O^*(c^s)$ running times *improve upon* those in the previous best known $O^*(c^W)$ running times for our problems. We complement these results by developing algorithms for Weighted Vertex Cover and Weighted Edge Dominating Set parameterized by $t \leq s$, the minimum size of a solution (for the unweighted version).

1.1. Previous work

Our problems are well known in graph theory and combinatorial optimization. They were also extensively studied in the area of parameterized complexity. We mention below known FPT results for their unweighted and weighted variants, parameterized by t and W, respectively.

Vertex Cover: VC is one of the first problems shown to be FPT. In the past two decades, it enjoyed a race towards obtaining the fastest FPT algorithm [4–9,3,10–12]. The best FPT algorithm, due to Chen et al. [12], has running time $O^*(1.274^t)$. In a similar race, focusing on graphs of bounded degree 3 [9,13–17], the current winner is an algorithm of Issac et al. [17], whose running time is $O^*(1.153^t)$. For WVC, Niedermeier et al. [3] proposed an algorithm of $O^*(1.396^W)$ time and polynomial space, and an algorithm of $O^*(1.379^W)$ time and $O^*(1.363^W)$ space. Subsequently, Fomin et al. [18] presented an algorithm of $O^*(1.357^W)$ time and space. An alternative algorithm, using $O^*(1.381^W)$ time and $O^*(1.26^W)$ space, is given in [19].

3-Hitting Set: Several papers study FPT algorithms for 3HS [20–23]. The best such algorithm, by Wahlström [23], has running time $O^*(2.076^t)$. For W3HS, Fernau [24] gave an algorithm which runs in time $O^*(2.247^W)$ and uses polynomial space.

Edge Dominating Set: FPT algorithms for EDS are given in [25–27], and the papers [28,29] present such algorithms for graphs of bounded degree 3. The best known algorithm for general graphs, due to Xiao et al. [27], has running time $O^*(2.315^t)$, and for graphs of bounded degree 3, the current best algorithm, due to Xiao et al. [29], has running time $O^*(2.148^t)$. FPT algorithms for WEDS are given in [25,19,26]. The best algorithm, due to Binkele-Raible et al. [26], runs in time $O^*(2.382^W)$ and uses polynomial space.

Max Internal Out-Branching: Although FPT algorithms for minimization problems are more common than those for maximization problems (see [30]), IOB was extensively studied in this area [31–39]. The previous best algorithms run in time $O^*(6.855^t)$ [39], and in randomized time $O^*(4^t)$ [34,38]. The weighted version, WIOB, was studied in the area of approximation algorithms (see [40,41]); however, to the best of our knowledge, its parameterized complexity is studied here for the first time.

We note that well-known tools, such as the color coding technique [42], can be used to obtain elegant FPT algorithms for some classic weighted graph problems (see, e.g., [43–45]). Recently, Cygan et al. [46] introduced a novel form of tree-decomposition to develop an FPT algorithm for minimum weighted graph bisection. Yet, for many other problems, including those studied in this paper, these tools are not known to be useful. We further elaborate in Section 3 on the limitations of known techniques in solving weighted graph problems.

1.2. Our results

We introduce a novel multivariate approach for solving weighted parameterized problems. Our framework yields fast algorithms whose running times are of the form $O^*(c^s)$. We demonstrate its usefulness for the following problems.

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