Journal of Computer and System Sciences ••• (••••) •••–•••

Contents lists available at ScienceDirect

Journal of Computer and System Sciences

www.elsevier.com/locate/jcss

Disjoint path covers with path length constraints in restricted hypercube-like graphs

Jung-Heum Park [a], Hee-Chul Kim [b], Hyeong-Seok Lim [c,*]

[a] School of Computer Science and Information Engineering, The Catholic University of Korea, Republic of Korea
[b] Division of Computer and Electronic Systems Engineering, Hankuk University of Foreign Studies, Republic of Korea
[c] School of Electronics and Computer Engineering, Chonnam National University, Republic of Korea

ARTICLE INFO

Article history:
Received 12 January 2016
Received in revised form 18 August 2016
Accepted 21 May 2017
Available online xxxx

Keywords:
Hypercube-like graph
Disjoint path cover
Path partition
Prescribed sources
Prescribed path lengths
Fault tolerance
Interconnection network

ABSTRACT

A *disjoint path cover* of a graph is a set of pairwise vertex-disjoint paths that altogether cover every vertex of the graph. In this paper, we prove that given k sources, $s_1, \ldots, s_k$, in an m-dimensional restricted hypercube-like graph with a set F of faults (vertices and/or edges), associated with k positive integers, $l_1, \ldots, l_k$, whose sum is equal to the number of fault-free vertices, there exists a disjoint path cover composed of k fault-free paths, each of whose paths starts at s_i and contains l_i vertices for $i \in \{1, \ldots, k\}$, provided $|F| + k \leq m - 1$. The bound, $m - 1$, on $|F| + k$ is the best possible.

1. Introduction

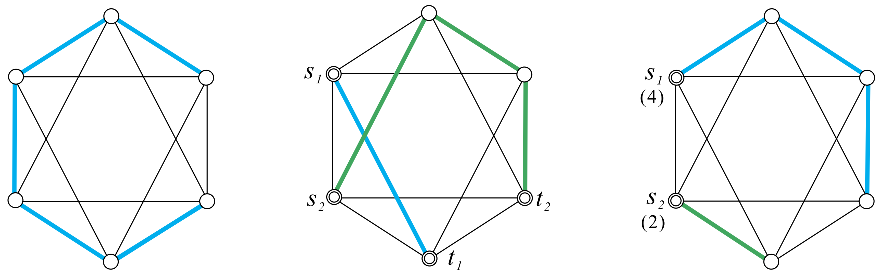
One of the central issues in the study of interconnection networks is the detection of parallel paths, which is naturally related to routing among nodes and fault tolerance of the network [18,28]. An interconnection network is frequently modeled as a graph, where vertices and edges represent the nodes and communication links of the network, respectively. Parallel paths correspond to pairwise disjoint paths of the graph. Disjoint path is, moreover, a fundamental notion from which many graph properties can be deduced [3,28].

A *Disjoint Path Cover* (*DPC* for short) of a graph is a set of pairwise disjoint paths that altogether cover every vertex of the graph. The disjoint path cover problem has applications in many areas such as software testing, database design, and code optimization [2,31]. In addition, the problem is concerned with applications where the full utilization of network nodes is important [37]. The disjoint path covers, with or without additional constraints on the paths, have been the subject of research for several decades. They can be categorized as several types, as discussed in the following, according to whether the terminals (sources and sinks) or the lengths of paths are prescribed or not. A path runs from its *source* to its *sink*.

Unconstrained DPCs. No prescribed terminals or lengths of paths are given in this type. The problem is to determine a disjoint path cover of a graph that uses the minimum number of paths. Fig. 1(a) shows an example of unconstrained DPCs. The minimum number is called the *path cover number* of the graph. Obviously, the path cover number of a graph is equal to one if and only if the graph contains a Hamiltonian path. Every hypercube-like graph [43] has a Hamiltonian path,

* Corresponding author.
E-mail addresses: j.h.park@catholic.ac.kr (J.-H. Park), hckim@hufs.ac.kr (H.-C. Kim), hslim@chonnam.ac.kr (H.-S. Lim).

http://dx.doi.org/10.1016/j.jcss.2017.05.007



(a) An unconstrained DPC

(b) A DPC with prescribed sources and sinks

(c) A DPC with prescribed sources and path lengths

Fig. 1. Three types of disjoint path covers.

and thus its path cover number is one. For more discussion on the disjoint path covers of this type, refer to, for example, [1,19,27,31,42].

DPCs with prescribed sources and sinks. Given pairwise disjoint terminal sets $S = \{s_1, \ldots, s_k\}$ and $T = \{t_1, \ldots, t_k\}$ of a graph, each representing k *sources* and *sinks*, the *many-to-many k-DPC* is a disjoint path cover each of whose paths joins a pair of source and sink. An example is shown in Fig. 1(b). There are two simpler variants: The *one-to-many k-DPC* for $S = \{s\}$ and $T = \{t_1, \ldots, t_k\}$ is a disjoint path cover made of k paths, each joining a pair of source s and sink t_i, $i \in \{1, \ldots, k\}$. The *one-to-one k-DPC* for $S = \{s\}$ and $T = \{t\}$ is a disjoint path cover in which each path joins an identical pair of source s and sink t. The disjoint path covers of this type have been studied for graphs such as hypercubes [5,6,9,10,16,21], recursive circulants [22,23], hypercube-like graphs [20,24,25,37,38], cubes of connected graphs [34], k-ary n-cubes [40,46], grid graphs [35], unit interval graphs [32], simple graphs [26], and DCell networks [44].

DPCs with prescribed sources and path lengths. Consider a given set of k sources, $S = \{s_1, \ldots, s_k\}$, in a graph G, associated with k positive integers, $l_1, \ldots, l_k$, such that $\sum_{i=1}^{k} l_i = |V(G)|$, where $V(G)$ denotes the vertex set of G. A *prescribed-source-and-length k-DPC* of G is a disjoint path cover composed of k paths, each of which is an s_i-path of length l_i for $i \in \{1, \ldots, k\}$, where s_i-path is a path starting at source s_i. See Fig. 1(c) for an example. Here, the *length* of a path refers to the number of vertices in the path. This type of disjoint path covers will be mainly discussed in this paper. From now on, a disjoint path cover whose type is not specified is assumed to be a prescribed-source-and-length type.

The problem of determining whether there exists a k-DPC in a general graph is NP-complete for any fixed $k \geq 1$, which can be reduced from the well-known HAMILTONIAN PATH problem in [15]. Research on prescribed-source-and-length type is relatively scarce, although some results for hypercubes can be found in [7,30]. Nebeský [30] proved that there exists a k-DPC in an m-dimensional hypercube, $m \geq 4$, for any set of k sources, $S = \{s_1, \ldots, s_k\}$, associated with k positive *even* integers, $l_1, \ldots, l_k$, whose sum is equal to 2^m, subject to $k \leq m - 1$. More general results established by Choudum et al. can be found in [7].

In this paper, we study the disjoint path cover problem for the class of Restricted Hypercube-Like graphs (RHL graphs for short) [36], which are a subset of nonbipartite hypercube-like graphs that have received considerable attention over recent decades. The class includes most nonbipartite hypercube-like networks found in the literature; for example, twisted cubes [17], crossed cubes [12], Möbius cubes [8], recursive circulants $G(2^m, 4)$ with odd m [33], multiply twisted cubes [11], Mcubes [41], and generalized twisted cubes [4]. The definition of RHL graphs is deferred to the next section (Definition 4). An m-dimensional RHL graph has 2^m vertices of degree m.

As node and/or link failure is inevitable in a large network, fault tolerance is essential to the network performance. Throughout this paper, let F denote a set of faults (vertices and/or edges) corresponding to the set of node and/or link failures. When a graph contains faulty elements, its k-disjoint path cover naturally means a k-disjoint path cover of the graph, $G \setminus F$, with the faults deleted.

Definition 1. A graph G is called *k-path partitionable* if G has a k-DPC for any set of k sources, $s_1, \ldots, s_k$, associated with any k positive integers, $l_1, \ldots, l_k$, such that $\sum_{i=1}^{k} l_i = |V(G)|$. A graph G is said to be *f_b-fault k-path partitionable* if $G \setminus F$ is k-path partitionable for any fault set F with $|F| \leq f_b$.

Our main result on the construction of disjoint path covers in RHL graphs can be stated as follows:

Theorem 1. *Every m-dimensional RHL graph is f_b-fault k-path partitionable for any $f_b \geq 0$ and $k \geq 1$ subject to $f_b + k \leq m - 1$, where $m \geq 3$.*

Note that the bound, $m - 1$, on $f_b + k$ in Theorem 1 is the maximum possible. (Suppose otherwise, no m-dimensional RHL graph would have a k-DPC when, for some vertex $v \in V(G) \setminus (S \cup F)$, its $m - k$ neighbors are vertex faults whereas its remaining k neighbors are sources, s_1 through s_k, with $l_i \neq 2$ for all i.)

Please cite this article in press as: J.-H. Park et al., Disjoint path covers with path length constraints in restricted hypercube-like graphs, J. Comput. Syst. Sci. (2017), http://dx.doi.org/10.1016/j.jcss.2017.05.007