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Decidability and universality of quasiminimal subshifts

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ABSTRACT

We introduce quasiminimal subshifts, subshifts having only finitely many subsystems. With $\mathbb{N}$-actions, their theory essentially reduces to the theory of minimal systems, but with $\mathbb{Z}$-actions, the class is much larger. We show many examples of such subshifts, and in particular construct a universal system with only a single proper subsystem, refuting a conjecture of [3].

1. Introduction

One of the most studied classes of subshifts[1] in the literature is that of minimal subshifts. These are precisely the nonempty subshifts containing no proper nonempty (sub-)subshifts. Some reasons that these subshifts are of great interest are that every subshift contains a minimal subshift, and many natural examples of subshifts, such as those generated by primitive substitutions and those generated by Toeplitz sequences, are minimal. We direct the reader to [1,2] for a discussion of such systems. More generally, dynamical systems on compact metric spaces always contain minimal subsystems.

In [3], zero-dimensional dynamical systems with an $\mathbb{N}$-action and an effective presentation, called *symbolic systems*, are studied from the point of view of computational universality. These systems generalize one-sided subshifts whose language is recursive, and also many other symbolic systems such as Turing machines, counter machines and tag systems. Given any finite clopen partition of a space and an ω-regular language of infinite words with partition elements as letters, the *model-checking problem* is defined as the problem of checking whether one of the sequences in the language corresponds to a sequence of observations along an orbit. The model-checking problem for regular languages is the question of whether a finite sequence of observations in the language can be made in the system. In [3] a system is called *decidable* if the model-checking problem for ω-regular languages is decidable, and *universal* if its model-checking problem for regular languages is Σ^0_1-complete (that is, the halting problem many-one reduces to it). The definitions imply that a universal system cannot be decidable,[2] but a gap is left between the two definitions, so that both decidability and undecidability results are maximally strong.

In the case of subshifts, the model-checking problem for regular languages amounts to asking whether the intersection of the language of the subshift with a given regular language is empty, and we use this as the definition, omitting the details of computable presentations. The main results of this paper are constructions of subshifts which are universal, and thus ω-regular languages do not play a major role in this paper, and we only discuss them for context.

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[1] Here, subshifts are sets $X \subset S^M$ which are topologically closed, and also closed under the shift action of M in the sense $X \cdot M \subset X$, where M is a monoid and S a finite set. In this paper, we consider the cases $M \in \{\mathbb{N}, \mathbb{Z}\}$.

[2] This is because the model-checking problem of regular languages many-one reduces to that for ω-regular languages.

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One of the main results of [3] is that if a minimal system is computable,[3] then it is decidable in the above sense. This shows in particular that many non-trivial things can be computed about minimal subshifts, as long as the forbidden patterns of the subshift can be enumerated. For example, given[4] a Turing machine enumerating the forbidden patterns of a (nonempty) minimal subshift X over the alphabet $\{0, 1, 2\}$, an algorithm can check whether there exists a subword w of a point of X where the number of 1s differs from the number of 0s by more than 7.

It is not particularly hard to show that if the forbidden words of X can be enumerated and X is minimal, then also the words that do occur in X can be enumerated (see Theorem 7), so that if such w exists, a rather simple algorithm can find it. The algorithm of [3] must be even smarter: if no w with the above properties exists, then after enumerating some finite number of forbidden patterns, the algorithm will state this fact.

The algorithm of [3] applies more generally to systems where every proper subsystem has nonempty interior, and to systems whose limit set is a finite union of minimal systems. It is interesting to ask where the precise border of decidability lies, by extending the family of subshifts further, and to this end the authors also make the following conjecture.

Conjecture 1 *([3]). A universal symbolic system has infinitely many minimal subsystems.*

Conversely, this conjecture would imply that a system with finitely many minimal subsystems has to be non-universal (even if not necessarily decidable). Note that this conjecture talks about infinitely many *minimal* subsystems, but allows us to have any number of subsystems in general. It turns out that this is not a very strong condition, and as such, the conjecture is false.

Proposition 1. *There exists a recursive universal subshift with finitely many minimal subsystems.*

The proof of this is given in Section 3, Proposition 3. Our example is a $\mathbb{Z}$-subshift, and we interpret subsystems in the sense of being closed under the $\mathbb{Z}$-action, while subsystems in the sense of [3] need to be closed under the induced $\mathbb{N}$-action only. It is easy to see that this does not change the number of minimal subsystems, although it may change the number of subsystems in general (see Proposition 4 for details), so that our example also provides an $\mathbb{N}$-system with the desired properties. Alternatively, one can directly modify our example to be one-sided.

The subshift is very simple, and we offer multiple variations of it. At its simplest, the number of minimal subsystems is one, and this subsystem is just a single point.[5] The subshift is countable, and in fact contained in a countable sofic shift. Alternatively, the subshift can be contained in a countable SFT, although with slightly more minimal subsystems. The Cantor–Bendixson rank of the enveloping countable sofic shift is 4. CB-rank 4, for a countable sofic shift, means that each point contains at most 3 disturbances to periodicity.[6] In addition to the subshift being very simple, also the universality is very strong: not only are intersections with regular languages Σ^0_1-complete, but even the *undirected halting problem*, the question of whether two given words u and v occur in the same point, is Σ^0_1-complete.

While X has only finitely many minimal subsystems, it has infinitely many subsystems altogether. In fact, the set of subshifts of X has the cardinality of the continuum (which is the maximal possible). In the abstract and introduction of [3], the authors state the conjecture in a weaker form, asking if a universal system should have infinitely many subsystems, without any mention of minimality. While this was presumably just meant as a shortened form of the statement of Conjecture 1, it is a natural next question whether at least this is true.

Question 1. Must universal systems have infinitely many subsystems?

We name this class for easier reference.

Definition 1. A subshift is *quasiminimal* if it has finitely many subshifts.

The question for us is then whether a recursive quasiminimal subshift can be universal. One might guess that if X has only finitely many subshifts, then it must be a quite simple extension[7] of the one or more minimal subshifts it necessarily contains, or perhaps even essentially just a union. It turns out that, unlike in Proposition 1, whether this is true depends on whether the acting monoid is $\mathbb{N}$ or $\mathbb{Z}$, that is, on whether our subshifts are one- or two-sided.[8]

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[3] In their paper, computable means recursive, or having $\Delta^0_1 = \Sigma^0_1 \cap \Pi^0_1$ language, but see Theorem 7.

[4] It is not explicitly stated in [3] that the algorithm is uniform in the description of the subshift, but this is clear from their proof.

[5] The one-point subshift is the simplest minimal subshift, but any minimal subshift can be used here, though naturally at the expense of countability.

[6] More precisely, there exists p such that every point x of the subshift satisfies $x_i = x_{i+p}$ for all $i \in \mathbb{Z} \setminus A$, where A is a union of three intervals which all have uniformly bounded length.

[7] Here, we use the term extension in the sense of containment (monomorphisms), and not in the sense of factoring (epimorphisms). This usage is very nonstandard in the theory of dynamical systems, but it is fitting for quasiminimal systems, since they are inductively built, in finitely many steps, from smaller quasiminimal systems by adding new points.

[8] Strictly speaking, we could also consider $S^{\mathbb{Z}}$ with an $\mathbb{N}$-action (obtaining a rather unnatural definition of a subshift), but the convention is that S^M uses the natural shift action of M. This is also required for the dynamical characterization of subshifts as expansive systems.

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