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On the parameterized complexity of b-CHROMATIC NUMBER ☆

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ABSTRACT

The b-chromatic number of a graph G, $\chi_b(G)$, is the largest integer k such that G has a k-vertex coloring with the property that each color class has a vertex which is adjacent to at least one vertex in each of the other color classes. In the b-Chromatic Number problem, the objective is to decide whether $\chi_b(G) \geq k$. Testing whether $\chi_b(G) = \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of a graph, itself is NP-complete even for connected bipartite graphs (Kratochvíl, Tuza and Voigt, WG 2002). We show that b-Chromatic Number is W[1]-hard when parameterized by k, resolving the open question posed by Havet and Sampaio (Algorithmica 2013). When $k = \Delta(G)+1$, we design an algorithm for b-Chromatic Number running in time $2^{\mathcal{O}(k^2 \log k)} n^{\mathcal{O}(1)}$. Finally, we show that b-Chromatic Number for an n-vertex graph can be solved in time $\mathcal{O}(3^n n^4 \log n)$.

1. Introduction

Graph coloring (proper vertex coloring), is an assignment of colors to the vertices of a graph such that no edge connects two identically colored vertices. In other words graph coloring is a partition of vertex set into independent sets. A proper vertex coloring using k colors is called a *k-vertex coloring*. The least number of colors required for a proper vertex coloring of a graph G is called the *chromatic number* of G. The most common question about graph coloring is – "what is the chromatic number of a graph". This question has received a lot of attention in graph theory and algorithms. The study of graph coloring led to the four color theorem in planar graphs by Appel and Haken [2], the study of chromatic polynomial introduced by Birkhoff, which was generalized to the Tutte polynomial by Tutte, and has inspired further graph-theoretic concepts. Graph coloring has been studied as an algorithmic problem since the early 1970s. The chromatic number problem is one of Karp's 21 NP-complete problems from 1972 [3]. An exact algorithm to compute the chromatic number of a graph dates back to 1976. Lawler [4] gave an algorithm for finding the chromatic number running in time $2.4423^n n^{\mathcal{O}(1)}$. Finally, after 30 years, using the principle of inclusion–exclusion Björklund et al. [5] gave an algorithm for the chromatic number problem running in time $2^n n^{\mathcal{O}(1)}$. This is still the fastest known exact algorithm to compute the chromatic number of a graph.

Not only finding the chromatic number but also different variations of graph coloring have been studied in the literature. A *complete coloring* of a graph G is a proper vertex coloring such that no two color classes together form an independent set. The parameter *achromatic number* of a graph G is the largest integer k such that there is a complete coloring of G using k colors. Irving and Manlove [6] introduced *b-chromatic number*, another parameter related to graph coloring.

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Definition 1.1 *(b-Chromatic number).* The *b-chromatic number* of a graph G, denoted by $\chi_b(G)$, is the largest integer k such that G has a k-vertex coloring with the property that each color class has a vertex which is adjacent to at least one vertex in each of the other color classes. Such a coloring is called a *b-coloring*.

Irving and Manlove showed that determining the b-chromatic number is NP-complete for general graphs, but polynomial time solvable for trees [6]. From the definition of the b-chromatic number it is clear that $\chi_b(G) \le \Delta(G)+1$, where $\Delta(G)$ is the maximum degree of the graph G. Kratochvíl et al. [7] showed that determining whether $\chi_b(G) = \Delta(G)+1$ is NP-hard even for connected bipartite graphs. Havet et al. [8] showed that the b-chromatic number can be computed in polynomial time for split graphs and it is NP-hard for connected chordal graphs. Regarding approximation algorithms for the problem, Galcík et al. [9] showed that the b-chromatic number of an n-vertex graph can not be approximated within a factor $n^{1/4-\epsilon}$ for any constant $\epsilon > 0$, in polynomial time, unless P=NP.

In this work we address the algorithmic question of the b-chromatic number in the realm of parameterized complexity and exact exponential time algorithms.

b-CHROMATIC NUMBER	**Parameter:** k
Input: An n-vertex graph G and an integer k	
Question: Is the b-chromatic number of G at least k	

In parameterized complexity the running time of an algorithm is measured in terms of multiple parameters of the input. A parameterized problem is a subset $\Pi \subseteq \Sigma^* \times \mathbb{N}$, where Σ is a finite alphabet and $\mathbb{N}$ is the set of natural numbers. We say that a parameterized problem Π is *fixed parameter tractable* (FPT) if there is an algorithm for the problem Π running in time $f(k)|x|^{\mathcal{O}(1)}$ on input (x,k), where f is an arbitrary function depending only on k and $|x|$ is the length of x. For a detailed overview of parameterized complexity reader is referred to monographs [10,11]. In the parameterized complexity framework, the b-CHROMATIC NUMBER problem is studied with a dual parameter by Havet et al. [12]. In particular, they show that one can decide whether $\chi_b(G) \ge n-k$ in time $2^{\mathcal{O}(k\log k)}n^{\mathcal{O}(1)}$ and asked the question whether b-CHROMATIC NUMBER is FPT when parameterized by k. Recently, Effantin et al. [13] studied a relaxed version of b-coloring and repeated the question about the parameterized complexity of b-CHROMATIC NUMBER. In this work we answer this question negatively, by showing that b-CHROMATIC NUMBER is W[1]-hard. But, when $k = \Delta(G)+1$, we design an algorithm for b-CHROMATIC NUMBER running in time $2^{\mathcal{O}(k^2\log k)}n^{\mathcal{O}(1)}$. Finally we show that b-CHROMATIC NUMBER for an n-vertex graph can be solved in time $\mathcal{O}(3^n n^4 \log n)$.

Our methods. To show b-CHROMATIC NUMBER is W[1]-hard, when parameterized by k, we give an FPT-reduction from MULTI-COLORED INDEPENDENT SET, which is very well known to be W[1]-hard [11]. When $k = \Delta(G)+1$, to get an FPT algorithm for b-CHROMATIC NUMBER, we first show that it is enough to find $C \subseteq V(G)$ such that $\chi_b(G[C]) = k$ (we call such a subset C as b-chromatic core of order k). Then we give a polynomial kernel for the problem of finding b-chromatic core of order $\Delta(G)+1$, which leads to an FPT algorithm for b-CHROMATIC NUMBER when $k = \Delta(G)+1$. For the exact exponential time algorithm for b-CHROMATIC NUMBER, we reduce the problem to many instances of single variate polynomial multiplication of degree 2^n.

2. Preliminaries

We use "graph" to denote simple graphs without self-loops, directions, or labels. We use $V(G)$, $E(G)$ and $\Delta(G)$, respectively, to denote the vertex set, edge set and maximum degree of a graph G. We also use $G = (V,E)$ to denote a graph G on vertex set V and edge set E. For $v, u \in V(G)$ and $V' \subseteq V(G)$, we use $G[V']$ to denote the subgraph of G induced on V', $N[v] = \{u \;:\; (v,u) \in E(G)\} \cup \{v\}$ and $d(u,v)$ is the shortest distance between u and v. For a graph G and a b-coloring of G with color classes $C_1, \ldots, C_k$, we say a vertex $v \in C_i$ is a *dominating* vertex (*dominator*) for the color class C_i if v is adjacent to a vertex in C_j for each $j \ne i$.

We use $[n]$ to denote the set $\{1, 2, \ldots, n\}$. We use $\uplus$ to denote the disjoint union of sets: for any two sets A, B, the set $A \uplus B$ is defined only if $(A \cap B) = \emptyset$, and in this case $(A \uplus B) = (A \cup B)$. We assume that $\uplus$ associates to the left; that is, we write $\biguplus_{1\le i\le n} A_i = A_1 \uplus A_2 \uplus A_3 \cdots \uplus A_n$ to mean $(\cdots((A_1 \uplus A_2) \uplus A_3) \cdots \uplus A_n)$. Further, every use of $\uplus$ in an expression carries with it the implicit assertion that the two sets involved are disjoint.

If A, B are binary strings, then by $A + B$ we mean the *integer* $val(A) + val(B)$ where for a binary string X the expression $val(X)$ denotes the integer of which X is a binary representation. Let $U = \{u_1, u_2, \ldots, u_n\}$ be a set of cardinality n, and let $S \subseteq U$. The *characteristic vector* $\chi(S)$ of S with respect to U is the binary string with $|U| = n$ bits whose ℓth bit, for $1 \le \ell \le n$, is 1 if element u_ℓ belongs to set S, and 0 otherwise. We use $\mathbb{N}$ to denote the set of non-negative integers. The *Hamming weight* $\mathcal{H}(r)$ of a binary string r is the number of 1s in r. For a finite set U, a subset $S \subseteq U$, and the characteristic vector $\chi(S)$ of S with respect to U, observe that $\mathcal{H}(\chi(S)) = |S|$. We define the Hamming weight of $n \in \mathbb{N}$ to be the number $\mathcal{H}(n)$ of the number of 1s in a binary representation of n. Note that an integer $n \in \mathbb{N}$ does not have a *unique* binary representation, since we can pad any such representation with zeroes on the left without changing its numerical value. We call the total number of bits in a binary representation r of n the *width* of r.