Journal of Computer and System Sciences 84 (2017) 288–301

Contents lists available at ScienceDirect

Journal of Computer and System Sciences

www.elsevier.com/locate/jcss

A small minimal aperiodic reversible Turing machine

Julien Cassaigne [a], Nicolas Ollinger [b,1], Rodrigo Torres-Avilés [c,*,1,2,3]

[a] Institut de Mathématiques de Marseille (UMR 7373), Université de Marseille, 13288 Marseille Cedex 9, France
[b] Univ. Orléans, INSA Centre Val de Loire, LIFO EA 4022, FR-45067 Orléans, France
[c] Departamento de Ciencias de la Computación y Tecnologías de la Información, Universidad del Bío-Bío, Chillán, Chile

ARTICLE INFO

Article history:
Received 11 May 2014
Received in revised form 28 September 2016
Accepted 3 October 2016
Available online 19 October 2016

Keywords:
Turing machines
Discrete dynamical systems
Symbolic dynamics

ABSTRACT

A simple reversible Turing machine with four states, three symbols and no halting configuration is constructed that has no periodic orbit, simplifying a construction by Blondel, Cassaigne and Nichitiu and positively answering a conjecture by Kari and Ollinger. The constructed machine has other interesting properties: it is symmetric both for space and time and has a topologically minimal associated dynamical system whose column shift is associated to a substitution. Using a particular embedding technique of an arbitrary reversible Turing machine into the one presented, it is proven that the problem of determining if a given reversible Turing machine without halting state has a periodic orbit is undecidable.

The present article is mainly devoted to show a particular Turing machine without halting configuration nicknamed "SMART" and to establish its nice dynamical properties:

- reversible,
- both space and time symmetric,
- without periodic points (aperiodic),
- topologically transitive and minimal,
- with a substitutive trace shift, and
- small: four states and three symbols,

Almost periodic points exists in any metric dynamical system [1], but several examples of systems without periodic points exist. Nevertheless, when in 1997 Kůrka considered Turing machines from a dynamical systems point of view, he conjectured that the existence of periodic points in this model was necessary. The intuition suggests that a non-trivial Turing machine will always need to cover arbitrary long distances to avoid temporal periodicity, and most machines do this by regularly moving over periodic configurations. A counterexample was found later by Blondel, Cassaigne and Nichitiu [2] and the approach taken by Kůrka gave birth to a series of publications about dynamical properties of Turing machines [3–6].

* Corresponding author.
E-mail addresses: julien.cassaigne@univ-mrs.fr (J. Cassaigne), nicolas.ollinger@univ-orleans.fr (N. Ollinger), rtorres@ubiobio.cl (R. Torres-Avilés).

1 This work has been supported by ECOS Sud – CONICYT project C12E05.
2 This work has been supported by Universidad del Bío-Bío, DIUBB 167419/FE.
3 This work has been supported by CONICYT BASAL project, CMM, Universidad de Chile, and CI^{2}MA, Universidad de Concepción and by FONDECYT #1140684.

http://dx.doi.org/10.1016/j.jcss.2016.10.004

Considering Turing machines as dynamical systems has several motivations (and difficulties). Motivations are related to the study of *complexity* of dynamical systems in the sense of predictability of the system behavior from its description and initial conditions. If a universal Turing machine is embedded into a given dynamical system, this one inherits the complexity of the machine, and in particular several questions about the system result to be undecidable. But typical undecidable problems about Turing machines are not always natural for dynamical systems. One of the drawbacks comes from the existence, in the context of Turing machines, of a *finite* initial configuration and a starting and a halting state. Which, in terms of dynamical systems, imply to work over a particular subset of points (usually a *non-compact* subset), and to be limited to systems that *halt* at some points (in particular when starting at the halting state). Kůrka looked at Turing machines directly as dynamical systems by omitting the halting and starting states and eliminating the notion of blank state to work over arbitrary initial tapes.

In this new context, the difficulty comes from the fact that very few undecidable problems consider the behavior of machines without initial state and starting from potentially infinite configurations. Surprisingly enough, the first problem of this kind was studied and proved undecidable in 1966 by Hooper [7]. The *immortality* problem asks for the existence of initial configurations where the machine does not halt (or where a given state is never attained). We remark that it is different from the *totality* problem, that asks for the existence of a *finite* configuration where the machine does not halt (for example, a machine that goes to the left over any symbol but halts when attaining the blank symbol is total, but not mortal). Defining mortal machines requires to avoid *unbounded searches* through a repetitive movement which would lead to immortal trajectories. Hooper uses a technique of *recursive calls* which implements unbounded searches by performing imbricate calls to searches of length three. This is the technique used in [2] to define a *complete* (without halting states) Turing machine without periodic points. There, the undecidability of the problem of existence of periodic points for complete Turing machines is also proved.

Another achievement in this direction is the undecidability of the periodicity problem, established by Kari and Ollinger [8]. They focus on the restricted class of *reversible* machines. It is not necessary to recall the importance of "reversibility" in the context of computer science and dynamical systems, but we would like to point out its relation to other properties. A system is called "reversible" when the precedent configuration is uniquely defined (if it exists). In the case of complete Turing machines, it results to be equivalent to bijectivity and surjectivity of the global transition function, and it can be easily checked from the machine transition rule. Surjectivity is a necessary condition for *transitivity*, which says that, for every pair of points, there is a third point that passes as close as we want to both. A system which is transitive has a high uniformity, since most points pass near every other point and have, then, a similar behavior over an arbitrary long time.

Minimality is a stronger property. A minimal system is a system without proper subsystems, *i.e.*, with no topologically closed subsets which are also closed for the dynamics. In a minimal system, every point passes, as close as we want, to any other point. Minimal systems have no periodic orbits, because these are proper subsystems. Thus aperiodic and reversible machines are good candidates to be minimal systems. The machines defined in [2] are not reversible, and the machines defined in [8] are not complete, thus no dynamical system can be defined from them. The methods used in these papers were not directly adaptable to the construction of such machines, and their existence was stated as an open question in [8].

The SMART machine answers this question positively and the associated system also results to be minimal. Usually minimal systems come from *substitutions* and SMART is not the exception. In Section 2 a subshift is associated to SMART, following [5], and it is proved that it is a *substitutive subshift* of a primitive substitution of non-constant length that we exhibit. Stronger than reversibility, time-symmetry says that the inverse of a function is conjugated with the function itself through an involution, which in this context means that the reverse of the Turing machine has the same rule that the direct machine, up to a transposition of the symbols and states. SMART is also time symmetric.

The existence of SMART allows the definition of a big family of aperiodic machines in Section 3, from which the problem of existence of periodic points, restricted to the context of reversible and complete Turing Machines, is proved to be undecidable.

1. A small aperiodic complete and reversible Turing machine

In 2002, Blondel, Cassaigne and Nichitiu [2] constructed the first aperiodic machine by using a technique inspired in the *Hooper calls* [7], which roughly consist into avoid the creation of infinite loops by using the tape as a stack. The resulting machine was aperiodic but not reversible.

Years later in 2008, Kari and Ollinger [8] conjectured the existence of an aperiodic, complete and *reversible* Turing machine, asking Cassaigne about it. The resulted machine, a modification of the one presented in [2], is the machine that we call SMART, whose aperiodicity is proved in this section. The proof mainly follows the scheme developed in [2].

1.1. Preliminary definitions and the machine

1.1.1. Word notation

In this work, Σ^* denotes the set of finite sequences of elements of Σ, called *finite words*. Also, Σ^ω represents the set of right-infinite sequences of elements of Σ, called *semi-infinite words*, and ${}^\omega\Sigma$ denotes the left-infinite sequences of Σ. Finally, $\Sigma^{\mathbb{Z}}$ denotes the set of *bi-infinite words*.