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Rural postman parameterized by the number of components of required edges

Gregory Gutin [a,*], Magnus Wahlström [a], Anders Yeo [b,c]

[a] Department of Computer Science, Royal Holloway, University of London, Egham, Surrey TW20 0EX, UK
[b] Department of Mathematics and Computer Science, University of Southern Denmark, Campusvej 55, 5230, Odense M, Denmark
[c] Department of Mathematics, University of Johannesburg, Auckland Park, 2006, South Africa

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ABSTRACT

In the Directed Rural Postman Problem (DRPP), given a strongly connected directed multigraph $D = (V, A)$ with nonnegative integral weights on the arcs, a subset R of required arcs and a nonnegative integer ℓ, decide whether D has a closed directed walk containing every arc of R and of weight at most ℓ. Let k be the number of weakly connected components in the subgraph of D induced by R. Sorge et al. [30] asked whether the DRPP is fixed-parameter tractable (FPT) when parameterized by k, i.e., whether there is an algorithm of running time $O^*(f(k))$ where f is a function of k only and the O^* notation suppresses polynomial factors. Using an algebraic approach, we prove that DRPP has a randomized algorithm of running time $O^*(2^k)$ when ℓ is bounded by a polynomial in the number of vertices in D. The same result holds for the undirected version of DRPP.

1. Introduction

In this paper, all walks in directed multigraphs (and their special types: trails, paths and cycles) are directed. For directed multigraphs, we mainly follow terminology and notation of [2]. A walk W is *closed* if the initial and terminal vertices of W coincide. A *trail* is a walk without repetition of arcs; a *path* is a trail without repetition of vertices; a *cycle* is a closed trail with no repeated vertices apart from the initial and terminal ones. A directed multigraph G is *weakly connected* (*strongly connected*, respectively) if there is a path between any pair of vertices in the underlying undirected graph of G (there are paths in both directions between any pair of vertices of G, respectively). A *weakly connected component* of G is a maximal weakly connected induced subgraph of G.

A closed trail in directed or undirected graph G is called *Eulerian* if it includes all edges and vertices of G; a graph containing an Eulerian trail, is called Eulerian. The *balance* of a vertex v of a directed multigraph H is the in-degree of v minus the out-degree of v. It is well-known that an undirected (directed, respectively) multigraph G is Eulerian if it is connected and each vertex is of even degree (weakly connected and the balance of every vertex is zero, respectively) [2,25]. Note that every Eulerian directed multigraph is strongly connected. For directed multigraphs, we will often use the term *connected* instead of *weakly connected*.

* Corresponding author.
E-mail addresses: gutin@cs.rhul.ac.uk (G. Gutin), Magnus.Wahlstrom@rhul.ac.uk (M. Wahlström), andersyeo@gmail.com (A. Yeo).

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The Chinese Postman Problem (CPP) can be formulated as follows: given a connected multigraph G with nonnegative integral weights on the edges, find a closed walk of minimum total weight which contains each edge of G at least once. CPP for both directed and undirected multigraphs is polynomial time solvable [25].

In this paper, we study the following generalization of Directed CPP:

> Directed Rural Postman Problem (DRPP)
> *Input:* A strongly connected directed multigraph $D=(V,A)$, a submultiset R of arcs of D, a weight function $\omega : A \to \mathbb{N}$, and an integer ℓ.
> *Question:* Is there a closed walk on D containing every arc of R with the total weight at most $\omega(R)+\ell$, where $\omega(R)$ is the total weight of arcs in R?

Note that if D is not strongly connected, DRPP has no feasible solution unless all arcs of R belong to the same strongly connected component D' of D, in which case we can replace D by D'.

We also study the Undirected Rural Postman Problem (URPP), where D is a connected undirected multigraph.

Practical applications of RPP include garbage collection, mail delivery and snow removal [1,10,11]. Both undirected and directed cases of RPP are NP-hard by a reduction from the Hamilton Cycle Problem [20] (see also [3]).

We will study the parameterized complexity of DRPP. A parameterized problem $\Pi \subseteq \Sigma^* \times \mathbb{N}$ is called *fixed-parameter tractable (FPT)* with respect to a parameter k if $(x,k) \in \Pi$ can be decided by an algorithm of running time $f(k)|x|^{O(1)}$, where f is a function only depending on k. (For background and terminology on parameterized complexity we refer the reader to the monographs [9,12,24].)

Consider DRPP and let k be the number of weakly connected components of $D[R]$, where $D[R]$ is the subgraph of D induced by R. Sorge et al. [30] noted that the complexity of DRPP parameterized by k "is a more than thirty years open ... question with significant practical relevance." Sorge et al. [30] commented that "k is presumably small in a number of applications [13,14]" and Sorge [28] remarked that in planning for snow plowing routes for Berliner Stadtreinigung, k is between 3 and 5. Lately, the question whether DRPP parameterized by k is FPT was raised in [3,8,18,28,29].

Frederickson [13,14] obtained a polynomial-time algorithm for DRPP when k is constant. However, k influences the degree of the polynomial in the running time of Frederickson's algorithm. Dorn et al. [8] proved that the DRPP is FPT when parameterized by the number a of arcs not from R in a solution of the problem. However, $k \le a$ and according to Sorge et al. [30] "it is reasonable to assume that k is much smaller [than a] in practice". Sorge et al. [29] proved that the DRPP is FPT when parameterized by $k+b$, where b is the sum of the absolute values of the balances of vertices in $G[R]$.

In the next section, we will prove that DRPP parameterized by k admits a randomized algorithm of running time $O^*(2^k)$ provided ℓ is bounded by a polynomial in the number of vertices in D. In fact, we prove this result for another problem called Eulerian Extension which is equivalent to DRPP.

In many applications of DRPP the weights are bounded by a polynomial in the number of vertices in the input digraph and so our result can be applied. Indeed, suppose the weight of every arc in D is bounded by a polynomial in the number n of vertices in D. Observe that the minimum weight solution of DRPP on D is bounded by the minimum weight solution of CPP on D, which is bounded by a polynomial in n. So, in this case, we may assume that ℓ is bounded by a polynomial in n.

Consider the following two examples. Höhn et al. [16] introduced the following problem equivalent to a problem in scheduling and proved that the problem is NP-complete. Given a directed multigraph $D=(V,A)$ with vertices $V \subset \mathbb{R}_0^+ \times \mathbb{R}_0^+$, determine whether there exists a collection H of pairs of vertices of the type (u,v) with $u_1 \ge v_1$ and $u_2 \ge v_2$, where $u=(u_1,u_2)$, $v=(v_1,v_2)$, such that $D+H$ is an Euler directed multigraph. Clearly, this problem is a special case of DRPP with all arcs being of weight 0 and 1: set $R=A$, assign weight 0 to pairs of vertices that we can add to H and weight 1 to all other pairs of vertices in V.

Golovnev et al. [15] obtained a reduction from the Shortest Common Superstring problem to DRPP parameterized by k and designed an $O^*(\gamma^n)$-time $(\gamma < 2)$ algorithm for Shortest Common Superstring with bounded length strings using our main result, Theorem 3. Note that the existence of such an algorithm for the general case of Shortest Common Superstring remains an open problem.

Sorge et al. [30] remarked that the complexity question "extends to the presumably harder undirected case of Rural Postman." We show that the DRPP algorithmic result holds also for URPP.

Henceforth, for a positive integer t, $[t]$ will stand for $\{1,\dots,t\}$. In an attempt to solve DRPP, Sorge et al. [30] introduced and extensively studied the following NP-hard matching problem.

> Conjoining Bipartite Matching (CBM)
> *Input:* A bipartite graph B with nonnegative weights on its edges, a partition $V_1 \cup \ldots \cup V_t$ of vertices of B, a number ℓ, and a graph $([t],F)$.
> *Question:* Decide whether B has a perfect matching M of total weight at most ℓ, such that for each $ij \in F$ there is an edge in M with one end-vertex in V_i and the other in V_j.