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Complexity of metric dimension on planar graphs ☆

Josep Diaz [a,b,1], Olli Pottonen [a,2,3], Maria Serna [a,b,1], Erik Jan van Leeuwen [c,*,4]

[a] *Department of Computer Science, UPC, Jordi Girona Salgado 1-3, 08034 Barcelona, Spain*
[b] *Barcelona Graduate School of Mathematics, UPC, Pau Gargallo 5, 08028 Barcelona, Spain*
[c] *Max-Planck Institut für Informatik, Campus E1 4, 66123 Saarbrücken, Germany*

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ABSTRACT

The metric dimension of a graph G is the size of a smallest subset $L \subseteq V(G)$ such that for any $x, y \in V(G)$ with $x \neq y$ there is a $z \in L$ such that the graph distance between x and z differs from the graph distance between y and z. Even though this notion has been part of the literature for almost 40 years, prior to our work the computational complexity of determining the metric dimension of a graph was still very unclear. In this paper, we show tight complexity boundaries for the Metric Dimension problem. We achieve this by giving two complementary results. First, we show that the Metric Dimension problem on planar graphs of maximum degree 6 is NP-complete. Then, we give a polynomial-time algorithm for determining the metric dimension of outerplanar graphs.

1. Introduction

In this paper, we study the complexity of the Metric Dimension problem, in particular on planar graphs. To define the Metric Dimension problem, we need several supporting notions. Let G be a graph. We say that $z \in V(G)$ *resolves* two vertices $x, y \in V(G)$ with $x \neq y$ if the length of a shortest path in G from z to x is different from the length of a shortest path in G from z to y. Then a set $L \subseteq V(G)$ is called a *resolving set* (or *metric generator*) of G if every pair $x, y \in V(G)$ with $x \neq y$ is resolved by some $z \in L$. We sometimes refer to the elements of a resolving set (or in fact, of any set of vertices that we hope to extend to a resolving set) as *landmarks*. Now the *metric dimension* of G is the cardinality of a smallest resolving set of G (such a smallest resolving set is known as a *metric basis*). The problem of determining the metric dimension of a given graph G is called Metric Dimension, but is also known as *Harary's problem* or the *rigidity problem*. The problem was defined independently by Harary and Melter [21] and Slater [29].

There are several reasons for studying the Metric Dimension problem. The first reason is that, even though the problem is part of Garey and Johnson's book on computational intractability [20], very little is known about the computational complexity of this problem. Garey and Johnson proved thirty years ago that the decision version of Metric Dimension is NP-

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* Corresponding author.
E-mail addresses: diaz@cs.upc.edu (J. Diaz), olli.pottonen@iki.fi (O. Pottonen), mjserna@cs.upc.edu (M. Serna), erikjan@mpi-inf.mpg.de (E.J. van Leeuwen).

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complete on general graphs [26] (another proof appears in [27]). Also it was shown that there exists a $2\log n$-approximation algorithm on arbitrary graphs [27], which is best possible within a constant factor under reasonable complexity assumptions [3,23]. Hauptmann et al. [23] showed hardness of approximation on sparse graphs and on complements of sparse graphs. On the positive side, fifteen years ago, Khuller et al. [27] gave a linear-time algorithm to compute the metric dimension of a tree (see also [29,21]), as well as a characterization for graphs with metric dimension 1 and several interesting properties of graphs with metric dimension 2. Similar results were independently obtained by Chartrand et al. [8]. Before we published a preprint of our work, no further results were known about the complexity of this problem. It is thus interesting if the substantial, long-standing gap on the tractability of this problem (between trees and general graphs) can be bridged.

After a preprint of our work appeared, a large number of papers have appeared that further investigate the complexity of Metric Dimension on graph classes. On the negative side, Epstein et al. [13] provided NP-hardness results for split graphs, bipartite graphs, co-bipartite graphs, and line graphs of bipartite graphs. Hoffman and Wanke [24], based on the NP-hardness reduction for planar graphs given in this paper, were able to prove that the problem is NP-hard on Gabriel unit disk graphs. More recently, Foucaud et al. [18,19] showed that the problem is NP-hard on permutation graphs and interval graphs. Fernau and Rodríguez-Velázquez [15] showed that on general graphs there is no algorithm running in $O(|V(G)|^{O(1)}2^{o(|V(G)|)})$ time unless the Exponential Time Hypothesis fails; this complements their algorithm running in $O(|V(G)|^{O(1)}2^{|V(G)|})$ time. Hartung and Nichterlein [22] settled the parameterized complexity for the standard parameter (the size of the resolving set) on general graphs, by showing that the problem is W[2]-complete even if the maximum degree is at most three; they also give a strong approximation hardness result on such graphs.

On the positive side, Epstein et al. [13] presented polynomial-time algorithms for a weighted variant of Metric Dimension on several graphs including paths, trees, and cographs. Fernau et al. [14] gave a polynomial-time algorithm for Metric Dimension on chain graphs, a subclass of bipartite graphs. Foucaud et al. [18,19] showed that Metric Dimension is fixed-parameter tractable for the standard parameter on interval graphs. Belmonte et al. [4] generalized this result to graphs of bounded treelength, which include not only interval graphs, but also chordal graphs, permutation graphs, and AT-free graphs.

The second reason for studying Metric Dimension is that the problem has received a lot of attention from researchers in different disciplines, in particular as a difficult graph theoretical problem (see e.g. [1,6,8,23] and references therein). For instance, a recent survey by Bailey and Cameron [1] notes an interesting connection to group theory and graph isomorphism. It was also shown to be applicable to certain cop-and-robber games [7] and to routing in networks [17]. Therefore it makes sense to continue the investigation on the computational complexity of Metric Dimension and narrow the above-mentioned complexity gap.

The third reason for studying Metric Dimension, particularly on planar and outerplanar graphs, is that known techniques in the area do not seem to apply to it. Crucially, it seems difficult to formulate the problem as an MSOL-formula, without which we cannot apply Courcelle's Theorem [9] on graphs of bounded treewidth. Hence, there is no easy way to show that the problem is polynomial-time solvable on graphs of bounded treewidth. Also, the line of research pioneered by Baker [2], which culminated in the recent meta-theorems on planar graphs using the framework of bidimensionality [11,16], does not apply, as Metric Dimension does not exhibit the required behavior. For example, the metric dimension of a (two-dimensional) grid is two [27] (see also [6]), whereas bidimensionality requires it to be roughly linear in the size of the grid. Moreover, the problem is not closed under contraction. This behavior of Metric Dimension contrasts that of many other problems, even that of other nonlocal problems such as Feedback Vertex Set. Hence, by studying the Metric Dimension problem, there is an opportunity to extend the toolkit that is available to us on planar graphs.

Our results In the present work, we significantly narrow the tractability gap of Metric Dimension. From the hardness side, we show that Metric Dimension on planar graphs, called Planar Metric Dimension, is NP-hard, even for planar graphs of maximum degree 6. From the algorithmic side, we show that there is a polynomial-time algorithm to find the metric dimension of outerplanar graphs.

The crux to both of these results is our ability to deal with the fact that the Metric Dimension problem is extremely *nonlocal*. In particular, a landmark can resolve vertices that are very far away from it. The paper thus focusses on constraining the effects of a landmark to a small area. The NP-hardness proof does this by constructing a specific family of planar graphs for which Metric Dimension is essentially a local problem. The algorithm on outerplanar graphs uses a tree structure to traverse the graph, together with several data structures that track the influence of landmarks on other vertices. As we show later, this is sufficient to keep the nonlocality of the problem in check. We believe that our algorithmic techniques are of independent interest, and could lead to (new) algorithms for a broad class of nonlocal problems.

Overview of the NP-hardness proof As a corollary of the work by Dahlhaus et al. [10], we prove a new version of Planar 3-SAT to be NP-complete. We reduce this problem to Metric Dimension. This is done by constructing a planar graph consisting of clause gadgets and variable gadgets. Let n be the number of variables. Each variable gadget must have four landmarks: three at known, specific locations, but for the fourth we have three different choices. They correspond to the variable being true, false, or undefined. These $4n$ landmarks are a resolving set if and only if they resolve all pairs of vertices in the clause gadgets, which happens only if they correspond to a satisfying truth assignment of the SAT-instance.