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An algebraic approach to multirelations and their properties

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ABSTRACT

We study operations and equational properties of multirelations, which have been used for modelling games, protocols, computations, contact, closure and topology. The operations and properties are expressed using sets, heterogeneous relation algebras and more general algebras for multirelations. We investigate the algebraic properties of a new composition operation based on the correspondence to predicate transformers, different ways to express reflexive–transitive closures of multirelations, numerous equational properties, how these properties are connected and their preservation by multirelational operations. We particularly aim to generalise results and properties from up-closed multirelations to arbitrary multirelations. This paper is an extended version of [7].

1. Introduction

A relation between two sets A and B is a subset of the Cartesian product $A \times B$. Reasoning about relations can be done using this set-based definition or, more abstractly, using (heterogeneous) relation algebras; for example, see [10,34,36]. One of the advantages of the algebraic approach is that many frequently used properties of relations can be expressed by concise equations or inequalities. For example, the relation R is transitive if and only if $RR \subseteq R$, using the composition of R with itself on the left-hand side of the inequality. In contrast, the usual set-based definition of transitivity involves three universally quantified variables. Because any inequality $Q \subseteq R$ can be translated to an equation $Q \cup R = R$ using the union of sets, we call such properties 'equational'. Similarly concise formulas can express properties of functions, orders, graphs and programs, which are often modelled by relations. Many examples can be found in [32,35].

In this paper we work towards a compendium of properties for multirelations. A multirelation is a relation between a set A and the powerset 2^B of a set B. The additional powerset structure is used, for example, for modelling two-player games, the interaction between agents in a computation, and the topological notion of a contact; for example, see [1,5,23,25]. Properties of multirelations appear in the literature typically in a set-based form. More recently, researchers have started to consider multirelations from an algebraic perspective, for example, in [18,19,21,22]. It is therefore a natural step to try to express multirelational properties algebraically, to find out how they are connected and which algebras are suitable for reasoning about them.

Studying properties of multirelations is not a straightforward generalisation of existing work on relations. Multirelations differ from arbitrary relations by using the powerset structure on their targets. As a consequence, they support operations – such as the multirelational dual – which are not available for arbitrary relations, and fail to support operations – such as converse – which are available for general relations. Unlike relational composition, which is a standard notion, at least four

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ways to sequentially compose multirelations have been studied in the literature; we will look at two of these in the present paper. Some of these composition operations fail to satisfy basic distributivity and associativity properties taken for granted for relational composition. While the algebra of relations has been studied at least since A. Tarski's axiomatisation 75 years ago in [36], the algebra of multirelations is much less well understood beyond its basic properties.

The properties considered in the present paper comprise several from the existing literature, others which are formally similar to properties of relations, properties obtained by formal dualisation, and further properties that turned out to be useful during our investigation. One of these properties describes up-closed multirelations, which are required in many previous works; a key achievement of the present paper is to show that many results about multirelations can be derived without a restriction to up-closed multirelations. Our method of study is mostly algebraic, in particular, as regards the relationships between properties as well as their preservation by multirelational operations. Many results in this paper have been verified using Isabelle/HOL and its integrated automated theorem provers and SMT solvers. Numerous counterexamples are provided to show that certain operations do not preserve certain properties; for the most part, these have been generated by a Haskell program. Besides the equational properties of multirelations, we study the algebraic properties of a new composition operation, relate various ways of describing reflexive–transitive closures of multirelations, and present a connection of multirelations to contact relations introduced by G. Aumann in [1].

An overview of this paper follows. In Sections 2 and 3 we start by representing multirelations and their operations in terms of relation algebras. We recall fundamental algebraic properties of multirelations and, in particular, of the composition operation introduced by R. Parikh in [25]. In Section 4 we show many algebraic properties of a different composition operation, which we have recently introduced based on a correspondence to predicate transformers in [8]. To further abstract from the relation-algebraic representation we introduce more general algebras in Section 5. They are based on Boolean algebras and semirings; their axioms capture fundamental properties of multirelational operations. In Section 6 we relate different definitions of reflexive–transitive closures in a very general algebraic setting which covers arbitrary multirelations. The above-mentioned equational properties of multirelations are studied in Section 7; in particular, we show numerous relationships between these properties. Section 8 is concerned with the preservation of these properties by multirelational operations. Our results here are complete in the sense that for each property and each fundamental operation we either prove that the property is preserved or falsify this by providing a counterexample. In Section 9 we discuss the connection to Aumann contact relations. Finally, Section 10 gives new logical characterisations of two distributivity properties of multirelations.

Overall, this paper introduces algebraic structures which capture arbitrary multirelations and uses these structures to study reflexive–transitive closure operations as well as equational properties of multirelations, their relationships and their preservation by multirelational operations. In addition, the paper gives logical representations of the properties and studies a composition operation recently introduced in [8]. In that companion paper we have investigated how some of the properties discussed here translate to predicate transformers. This was facilitated by a relation-algebraic correspondence between multirelations and predicate transformers, which is similar to the correspondence between contact relations and closure operations.

The contributions of this extended version with respect to the first version [7] are

1. a relation-algebraic investigation of the properties of an alternative composition operation of multirelations defined in [8] (Section 4);
2. additional properties of zero-vectors, one-vectors and down-closed multirelations, an investigation of their relationships with other properties and their preservation by multirelational operations, and the verification of the results in Isabelle/HOL (Sections 7.1 and 7.2 and Figs. 1, 2, 3 and 6);
3. an extension of algebraic structures for multirelations by a complement operation and an investigation of which properties it preserves, again verified in Isabelle/HOL (Section 7.2 and Fig. 7);
4. answers to the open questions in [7] regarding the preservation of $\cup$- and $\cap$-distributivity, and thereby a complete decision of which properties are preserved by which operations (Fig. 5 and Theorem 21);
5. characterisations of arbitrary $\cup$- and $\cap$-distributive multirelations including a new, weak finiteness condition, and derivation of two previous results as special cases (Section 10).

Most parts of the remaining Sections 2, 3, 5, 6, 8 and 9 are taken from the first version [7] with only small changes to reflect our new results.

2. Relation-algebraic prerequisites

In this section we present the facts about relations and heterogeneous relation algebras that are needed in the remainder of this paper. For more details on relations and relation algebras, see [32], for example.

We write $R : A \leftrightarrow B$ if R is a (typed binary) relation with source A and target B, that is, of type $A \leftrightarrow B$. If the sets A and B are finite, we may consider R as a Boolean matrix. Since this interpretation is well suited for many purposes, we will use matrix notation and write $R_{x,y}$ instead of $(x, y) \in R$ or $x\,R\,y$.

We assume the reader to be familiar with the basic operations on relations, namely R^{c} (converse), $\overline{R}$ (complement), $R \cup S$ (union), $R \cap S$ (intersection), RS (composition), the predicates $R \subseteq S$ (inclusion) and $R = S$ (equality) and the special rela-