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Algebraic solution of tropical optimization problems via matrix sparsification with application to scheduling

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ABSTRACT

Optimization problems are considered in the framework of tropical algebra to minimize and maximize a nonlinear objective function defined on vectors over an idempotent semifield, and calculated using multiplicative conjugate transposition. To find the minimum of the function, we first obtain a partial solution, which explicitly represents a subset of solution vectors. We characterize all solutions by a system of simultaneous equation and inequality, and show that the solution set is closed under vector addition and scalar multiplication. A matrix sparsification technique is proposed to extend the partial solution, and then to obtain a complete solution described as a family of subsets. We offer a backtracking procedure that generates all members of the family, and derive an explicit representation for the complete solution. As another result, we deduce a complete solution of the maximization problem, given in a compact vector form by the use of sparsified matrices. The results obtained are illustrated with illuminating examples and graphical representations. We apply the results to solve real-world problems drawn from project (machine) scheduling, and give numerical examples.

1. Introduction

Tropical (idempotent) mathematics focuses on the theory and applications of semirings with idempotent addition, and had its origin in the seminal works published in the 1960s by Pandit [29], Cuninghame-Green [6], Giffler [9], Hoffman [13], Vorob'ev [35], Romanovskiĭ [31], Korbut [15], and Peteanu [30]. An extensive study of tropical mathematics was motivated by real-world problems in various areas of operations research and computer science, including path analysis in graphs and networks [29,30], machine scheduling [6,9], production planning and control [31,35]. The significant progress in the field over the past few decades is reported in several research monographs, such as ones by Baccelli at al. [2], Kolokoltsov and Maslov [14], Golan [10], Heidergott et al. [12], McEneaney [27], Gondran and Minoux [11], Butkovič [3], Maclagan and Sturmfels [26] as well as in a wide range of contributed papers.

Since the early studies [9,13,30,31], optimization problems that can be examined in the framework of tropical mathematics have formed a notable research domain in the field. These problems are formulated to minimize or maximize functions defined on vectors over idempotent semifields (semirings with multiplicative inverses), and may involve constraints in the form of vector equations and inequalities. The objective functions can be both linear and nonlinear in the tropical mathematics setting.

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The span (range) vector seminorm, which is defined as the maximum deviation between components of a vector, presents one of the objective functions encountered in practice. Specifically, this seminorm can serve as the optimization criterion for just-in-time scheduling (see, e.g., Demeulemeester and Herroelen [8], Neumann et al. [28], T'kindt and Billaut [33] and Vanhoucke [34]), and finds applications in real-world problems that involve time synchronization in manufacturing, transportation networks, and parallel data processing.

In the context of tropical mathematics, the span seminorm has been introduced by Cuninghame-Green [5], and Cuninghame-Green and Butkovič [7]. The seminorm was used by Butkovič and Tam [4] and Tam [32] in optimization problems drawn from machine scheduling. A manufacturing system was considered, in which machines start and finish under some precedence constraints to make components for final products. The problems were to find the starting time for each machine to provide the completion times that are spread over either the shortest or longest time intervals. Solutions were given within a combined framework that involves two reciprocally dual idempotent semifields. Similar problems in the general setting of tropical mathematics were examined by Krivulin in [17,22,23], where direct, explicit solutions were suggested. However, the results obtained present a partial solution, rather than a complete solution, or offer a solution in scalar terms, rather than in a compact vector form.

We consider the tropical optimization problems formulated in [17,22,23] as extensions of the problems of minimizing and maximizing the span seminorm, and represent them in a slightly different form to

$$\text{minimize (maximize)} \quad \boldsymbol{q}^{-}\boldsymbol{x}(\boldsymbol{A}\boldsymbol{x})^{-}\boldsymbol{p},$$

where $\boldsymbol{p}$ and $\boldsymbol{q}$ are given vectors, $\boldsymbol{A}$ is a given matrix, $\boldsymbol{x}$ is the unknown vector. The minus sign in the superscript indicates multiplicative conjugate transposition of vectors, and the matrix–vector multiplications are thought of in the sense of tropical algebra.

The purpose of this paper is twofold. First, to obtain complete solutions to both minimization and maximization problems in an explicit vector form. We extend the partial solution of the minimization problem, which is obtained in [17] in the form of a subset of solution vectors, to a complete solution, describing all vectors that solve the problem. We combine the approach developed in [16,17,19,20,24] to reduce the problem to a system of simultaneous equation and inequality, with a new matrix sparsification technique to describe all solutions to the system. We use sparsified matrices to transform the complete solution of the maximization problem given in [22] into a compact vector form as well.

The second purpose is to apply the above results to the solution of real-world problems taken from just-in-time and scarce resource scheduling. We consider a project that involves a set of activities operating in parallel under temporal constraints imposed on the start and finish times of activities in the form of start–start, start–finish and finish–start precedence relations, and the finish deadline time boundaries. The problems are to minimize or maximize the maximum deviation of the finish times of activities, subject to the given constraints. These scheduling objectives reflect various possible resource limitations, such as manpower, energy and location constraints, which can require that all activities be finished simultaneously, or conversely, that the finish times be spread over the longest time interval.

We use results in [17–20], which enable to represent a range of scheduling problems as optimization problems in terms of tropical algebra, and then to obtain direct closed-form solutions to the problems on the basis of methods of tropical optimization. Note that existing solutions to the problems of interest generally present iterative algorithms that produce a solution if any exists, or indicate that there are no solutions (see, e.g., [8,28,33,34] for further details and comprehensive reviews). Moreover, many problems can be expressed as linear and mixed-integer linear programs, and then solved using an appropriate computational scheme of (mixed-)linear programming, which, in general, does not guarantee a direct solution in a closed form.

This paper further extends and supplements the results presented in the conference paper [21], which examined only the minimization problem, and focused on theoretical aspects of tropical optimization, rather than on applications of the results. Specifically, the current paper offers a new complete solution, obtained in a compact vector form by the use of sparsified matrices, to the maximization problem under study as well. In addition to the theoretical results, the paper describes, in detail, the application of the results to solve scheduling problems, and gives illuminating examples.

The solutions obtained for the scheduling problems under both minimization and maximization of the maximum deviation of finish times present quite new results. For instance, we derive a complete solution of the scheduling problem under the minimization criterion, which significantly extends previously known partial solutions [17,23]. The maximization problem under study generalizes those considered in [22] by taking into consideration additional constraints. Moreover, we offer a solution to the problem, which, in contrast to the scalar representation in [22], is given in a compact vector form, ready for further analysis and practical use.

We start with a brief overview of basic definitions, notation, and preliminary results of tropical mathematics in Section 2 to provide a general framework for the solutions in the later sections. Specifically, a lemma that offers two equivalent representations for a vector set is presented, which is of independent interest. Section 3 presents formulations for both minimization and maximization problems under consideration.

To solve the minimization problem in Section 4, we first find the minimum in the problem, and offer a partial solution in the form of an explicit representation of a subset of solution vectors. We characterize all solutions to the problem by a system of simultaneous equation and inequality, and exploit this characterization to investigate properties of the solutions. Furthermore, we develop a matrix sparsification technique, which consists in dropping entries below a prescribed threshold in the matrix of the problem without affecting the solution. By combining this technique with the above characterization,