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An exercise on the generation of many-valued dynamic logics

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ABSTRACT

In the last decades, dynamic logics have been used in different domains as a suitable formalism to reason about and specify a wide range of systems. On the other hand, logics with many-valued semantics are emerging as an interesting tool to handle devices and scenarios where uncertainty is a prime concern. This paper contributes towards the combination of these two aspects through the development of a method for the systematic construction of many-valued dynamic logics. Technically, the method is parameterised by an action lattice that defines both the computational paradigm and the truth space (corresponding to the underlying Kleene algebra and residuated lattices, respectively).

1. Introduction

1.1. Context

Propositions, capturing static properties of program states, and events, or actions, standing for state transitions, are the key ingredients in modelling and analysing about state-based software systems. Programs are typically combined through a Kleene algebra to express sequential, non deterministic, iterative behaviours, while propositions bring to the scene a logical structure.

Dynamic logic [17], a generalisation of the logic of Floyd–Hoare, is a well known and particularly powerful way of combining these two dimensions into a formal framework to reason about computational systems. Its potential stems from blending together classical logic, enriched with a modal dimension to express system's dynamics, and a (Kleene) algebra of actions to structure programs.

Over time dynamic logic grew to an entire family of logics increasingly popular in the verification of computational systems, and able to evolve and adapt to new, and complex validation challenges. One could mention its role in model validation (as in *e.g.* [26]), or the whole family of variants tailored to specific programming languages (as in *e.g.* [32,2]), or its important extensions to new computing domains, namely probabilistic [20] or continuous [38,39].

The latter is particularly relevant from an Engineering point of view: Actually, Platzer's hybrid dynamic logic, and its associated tool, KeYmaera, combining an algebra of actions based on real numbers assignments, with the standard Kleene operators and differential equations to specify continuous transitions from the "real" (physical) world, provides a powerful framework with increased industrial relevance for the design and validation of hybrid systems.

If hybrid systems entail the need to handle continuous state spaces, in a number of other cases dealing with some form of "quantitative" transitions (weighted, costed, probabilistic, certainty degrees etc.) is also a must. This motivates

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research to define dynamic logics over structures able to model weighted computations. On the logical side, expressing the validity of a formula through a Boolean outcome can be also be quite restrictive when dealing with complex, often unpredictable, systems. This motivates the adoption of logics with many-valued semantics, *e.g.* fuzzy [15], probabilistic [33] or weighted [10].

In such a context, this work attempts to combine dynamic logic and many-valued semantics to capture smoothly these kind of phenomena. The first steps in this direction appeared in a previous conference paper [29] where a generic, parametric, method to construct (propositional) many-valued dynamic logics was discussed. Technically, the definition of these logics is parameterised by an action lattice [21] which combines a Kleene algebra with a residuated lattice structure. This algebraic structure fits well our goal. On the one hand, as a residuated lattice, it provides an abstract structure for the truth spaces – most of the semantic structures used as truth spaces, such as *Boolean algebras*, *Heyting algebras*, *MV algebras* or *Łukasiewicz arithmetic lattices*, are residuated lattices (*e.g.*, [14,19]). Here, the residues take the role of logic implication. On the other hand as a Kleene algebra, it provides an abstract, generic, model for computations (*e.g.*, probabilistic, weighted,...). Moreover the extension of Kleene algebras with a residuated operator, providing a left inverse to sequential composition as in [40], as well as with a lattice structure, leads to a finitely-based equational variety which, as plain Kleene algebras, is closed under the formation of square matrices [22]. The relevance of this closure property lies in the fact that several problems modelled as (weighted) transition systems can be formulated as matrices over a Kleene algebra or a related structure. Following such a trend, we represent programs as matrices supporting the information about their effects when executed from each state in the state space.

1.2. Contributions

This paper is part of a research agenda on a systematic development of dynamic logics.[1] In particular, it extends preliminary results documented in [29] in several directions. On the one hand, the scope of the parameters is generalised by taking arbitrary action lattices instead of just the integral ones. The expressiveness of the dynamisation process is also strongly enriched, through the consideration of negations, [_]-modalities and tests. On the other hand, we took the challenge here to characterise, in some sense, "*how dynamic dynamisations are?*". The issue is addressed through an exercise: assuming the axiomatics of propositional dynamic logic (PDL) as a reference, we carry on a systematic study of the validity of some of its particular fragments with respect to particular dynamisation classes. Note, that this criteria is not an absolute reference to judge what is, and what is not, a dynamic logic. Although PDL is probably the most popular dynamic logic, others exist which do not satisfy such axiomatics. Such is the case, for example, of *game logic* [37].

To the best of our knowledge, beyond our preliminary work [29], the approaches reported in [18,25] are the unique references in the literature addressing many-valued dynamic logic. In the first paper J. Hughes *et al.* introduced a propositional dynamic logic over the continuum truth $(0, 1)$-lattice with the standard fuzzy residues (actually the ones adopted in Example 4). In particular, this logic can be achieved by weakening a specific instance built with the general construction introduced in the present paper. However, from the perspective of dynamic logic, this formalism is quite restrictive, since it lefts behind both transitive closure and non-deterministic choice.

In the context of rational decision theory, C. Liau [25] introduced a many-valued dynamic logic w.r.t. the specific continuum truth $(0, 1)$-lattice. Admitting some level of parametricity on the implication adopted (through a notion of implication function, of which the implications of Łukasiewicz and Gödel are examples), the semantics of [_]-modalities becomes quite different from what we get. By this reason, differently from what happens with the one introduced in [18], this logic can not be captured with our generic formalism.

More extensive work exist in the related field of many-valued modal logics. Two approaches are usually considered. The first one is clearly conservative, in the sense that the many-valued semantics only affects the modal valuation of propositions. In this case the accessibility relations are crispy (classic). The second one, closer to our own, considers that accessibility relations can themselves be many-valued. This approach was introduced by M. Fitting in [11,12], with many-valueness evaluated in finite Heyting algebras. Later it was deeply investigated by F. Bou *et al.* in [6], who adopted the more generic truth support of finite integral commutative residuated lattices. A middle-term between crispy and many-valued accessibility relations, appears in some works (*e.g.* [5,31]) through multi-modalities: for the cases where the truth lattice is a chain, any multi-valued relation can be equivalently expressed using a decreasing family of crispy modal relations, indexed by the support of the respective lattice.

1.3. A tribute to José Nuno Oliveira

The interplay between logic and computation, lying at the very heart of dynamic logic, is pervasive in the scientific work of José Nuno Oliveira. From his perspective, any computational phenomenon is an arrow in a suitable universe whose source and target are logic expressions. Thus, we find computations typed by invariants, as in a calculus of conductive programs [1], functions, in a calculus of data dependencies used for type checking database operations and query optimisation [36], or program assertions, in the form of coreflexive relations, as in a Hoare logic like calculus [34] targeting correct-by-construction program design, rather than verification.

[1] http://wiki.di.uminho.pt/twiki/bin/view/Research/Dali/WebHome.

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