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Higher dimensional Eisenstein–Jacobi networks

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HIGHLIGHTS

- Higher dimensional EJ network can be constructed based on lower dimensional EJ networks.
- The distance distribution of the nodes in the network is given.
- It is shown that higher dimensional EJ networks cost less and have more nodes than the GHC networks.
- The broadcasting algorithm for higher dimensional EJ network is discussed.
- A method of construction of edge disjoint Hamiltonian cycles is given with their Gray codes.

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ABSTRACT

An efficient interconnection topology called Eisenstein–Jacobi (EJ) network has been proposed in Martínez et al. (2008). In this paper this concept is generalized to higher dimensions. Important properties such as distance distribution and the decomposition of higher dimensional EJ networks into edge-disjoint Hamiltonian cycles are explored in this paper. In addition, an optimal shortest path routing algorithm and a one-to-all broadcast algorithm for higher dimensional EJ networks are given. Further, we give comparisons between higher EJ networks and Generalized Hypercube (GHC) networks and we show that higher EJ networks cost less and have more nodes than GHC networks.

1. Introduction

Hexagonal networks are modeled by planar graphs. Another network called honeycomb can also be seen as a hexagonal network. The duality between graphs corresponding to hexagonal and honeycomb networks causes some inconsistency in the name selection. There exist three regular plane tessellations: triangular, square, and hexagonal, which are the basis for the design of some interconnection networks. Hexagonal networks are six-degree based on regular triangular tessellations, whereas, honeycomb networks are three-degree based on regular hexagonal tessellations. Fig. 1 illustrates both networks, the solid lines represent the links of the hexagonal network and the dashed lines represent the links of the honeycomb network where the circles are the nodes of the corresponding network. Furthermore, mesh and tori networks are based on four-degree regular square tessellations.

There are many applications for hexagonal networks. For example, the University of Michigan developed the hexagonal torus in their HARTS project [30]. Further, hexagonal networks were used in image processing [28] to sampling images hexagonally rather than rectangularly, which gives higher angular resolution and more efficient sampling. They were also used in computer graphics [25], geological mapping [31], and cellular networks [27].

There exist three definitions for higher dimensional hexagonal networks. As discussed in [14], the first definition mentioned in [10] defines the 2D hexagonal network based on triangular tessellations. The 3D hexagonal network is built based on cubes of size t either limited by horizontal and vertical planes or limited by a diagonal plane. However, it was not a good approach since there is no formula for computing the distance between two nodes. In addition, in that network, the edges of the same length cannot be built. Also, the proposed 3D hexagonal mesh of size two should contain triangles, but they are both right angle and equilateral. Further, the second definition mentioned in [27,19],

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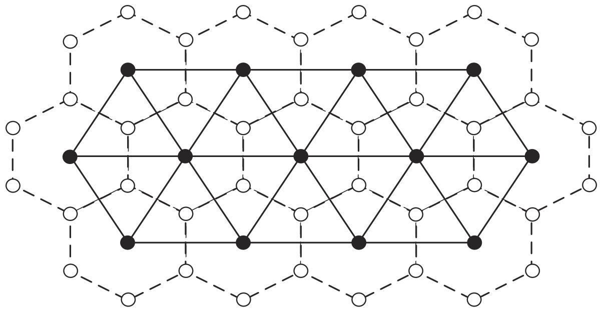

Fig. 1. Hexagonal and honeycomb networks.

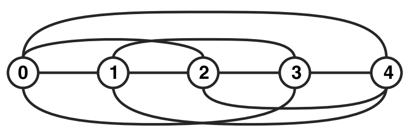

Fig. 2. Generalized Hypercube Q_5.

a 3D hexagonal mesh was modeled based on the concept of hypercube networks, which is not a natural generalization of 2D hexagonal mesh. The third definition was discussed in [14], which is a 3D hexagonal network based on the union of 2D hexagonal networks, but the authors did not go beyond the third dimension. In addition, their model is neither symmetric nor regular network, i.e., some nodes have different degree.

An efficient interconnection topology called Eisenstein–Jacobi (EJ) network has been proposed in [26]. An EJ network is mathematically defined and developed based on EJ integers. There are two apparent advantages of these networks: they are degree six symmetric networks and they are generalizations of the hexagonal mesh topology developed earlier in [11,16,23]. Another advantage of EJ networks is that they are used in graph theory as graphs that lead to a new method for constructing some classes of perfect codes, which are used to solve the problem of finding perfect dominating set [26,21]. Also, some applications of this network such as routing, broadcasting, and Hamiltonian cycles have been studied in [1,18,22].

The performance and cost of an interconnection network are based on some important characteristics such as the number of the nodes in the network; the degree of a node, which is the number of links per node; and the diameter of the network, which is known to be the maximum length of the shortest paths between any two nodes in the network and it corresponds to the worst communication time in the network. Usually, a network with higher degree has lower diameter than the network with lower degree. Further, the cost of the network is defined as diameter × degree.

In [29,9], the authors showed that higher dimensional Gaussian networks outperform the multi-dimensional toroidal networks. In this paper, we extend the EJ network beyond three dimensions and our approach is similar to the approach used in [29,9]. In addition, we show that the proposed network has lower cost than the Generalized Hypercube (GHC) [7]. Further, we give some communication algorithms such as shortest-path routing and broadcasting. We also show how to decompose this network into edge-disjoint Hamiltonian cycles.

The rest of the paper is organized as follows. Section 2 briefly reviews some literature that are relevant to the rest of the paper such as graph cross product, Generalized Hypercube networks, and topological properties of the original EJ networks. The definition and some basic topological properties of higher dimensional EJ networks are given in Section 3. Next, Section 4 provides a shortest-path routing algorithm for the network, while the distance distribution and the comparisons between higher EJ and GHC networks are given in Section 5. Section 6 shows how broadcasting can be done. Finding edge-disjoint Hamiltonian cycles is described in Section 7. The paper is concluded in Section 8.

2. Background

In this section, we review the cross product operation between two graphs since it is used in this paper. In addition, we give the definitions of Generalized Hypercube networks and EJ networks.

2.1. The cross product

The cross product between two graphs is defined as follows.

Definition 1 (*[15]*). The cross product between two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is denoted by $G_1 \otimes G_2$ and is defined as the graph $G(V, E)$, where

1. $V = \{(u, v) \mid u \in V_1, v \in V_2\}$
2. $E = \{((u_1, v_1), (u_2, v_2)) \mid ((u_1, u_2) \in E_1$ and $v_1 = v_2)$ or $((v_1, v_2) \in E_2$ and $u_1 = u_2)\}$.

For any finite set of networks $G_0, \ldots, G_n$, their cross product can be inductively defined as:

$$G_n \otimes G_{n-1} \otimes \cdots \otimes G_0 = G_n \otimes (G_{n-1} \otimes \cdots \otimes G_0).$$

This is used to define an n-dimensional Eisenstein–Jacobi network $EJ_\alpha^{(n)}$ in Section 3 as the n-fold cross product of the Eisenstein–Jacobi network EJ_α.

2.2. Generalized Hypercube networks

A single dimensional GHC network [7] Q_k, k is the size of the dimension, is a complete graph where all nodes are directly connected to each other. The degree of the network is $k-1$ and its diameter is 1. The nodes are addressed from 0 to $k-1$. Fig. 2 shows an example of Q_5.

The multi-dimensional GHC $Q_{k_{n-1} \times k_{n-1} \times \cdots \times k_0}$ is based on the cross products between single dimensional GHC networks and it is defined as follows.

$$Q_{k_{n-1} \times k_{n-1} \times \cdots \times k_0} = Q_{k_{n-1}} \otimes Q_{k_{n-2}} \otimes \cdots \otimes Q_0$$

where n is the number of dimensions, which is the diameter of the network; and k_i, for $0 \le i \le n-1$, is the size of the ith dimension. The node degree in the network is equal to $\sum_{i=0}^{n-1} k_i - 1$. Each nodes in the network is addressed using mixed radix number system and it is labeled as n-tuples, $(x_{n-1}, x_{n-2}, \ldots, x_0) \in \mathbb{Z}_{k_{n-1}} \times \mathbb{Z}_{k_{n-2}} \times \cdots \times \mathbb{Z}_{k_0}$. That is, $(x_{n-1}, x_{n-2}, \ldots, x_0) = x_{n-1}(k_{n-2} \times k_{n-3} \times \cdots \times k_0) + x_{n-2}(k_{n-3} \times k_{n-4} \times \cdots \times k_0) + \cdots + x_1 k_0 + x_0$. Each node $X = (x_{n-1}, x_{n-2}, \ldots, x_{i+1}, x_i, x_{i-1}, \ldots, x_1, x_0)$ is connected to the nodes $(x_{n-1}, x_{n-2}, \ldots, x_{i+1}, x_i', x_{i-1}, \ldots, x_1, x_0)$ for all of $0 \le i \le n-1$, where x_i' takes all integer values between 0 and $k_i - 1$ except x_i. Thus, two nodes $(x_{n-1}, x_{n-2}, \ldots, x_0)$ and $(y_{n-1}, y_{n-2}, \ldots, y_0)$ are neighbors if the Hamming distance between them is 1. The Hamming distance (D_H) between two n-tuples is the number of positions they differ. For example, $D_H((2214), (2234)) = 1$ and $D_H((3112), (2222)) = 3$. Fig. 3 illustrates an example of $Q_{5\times 5}$.

2.3. Eisenstein–Jacobi networks

EJ networks are designed based on the concept of EJ integers [21]. The set of *Eisenstein–Jacobi integers* $\mathbb{Z}[\rho]$ is defined as

$$\mathbb{Z}[\rho] = \{x + y\rho \mid x, y \in \mathbb{Z}\}$$