Theoretical Computer Science ••• (••••) •••–•••

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Garden of Eden partitions in the sand pile and related models

Jaclyn Anderson [a], Kristina C. Garrett [b], Brian Hopkins [c,*]

[a] Omaha, NE, United States
[b] St. Olaf College, United States
[c] Saint Peter's University, United States

ARTICLE INFO

Article history:
Received 29 June 2016
Accepted 28 June 2017
Available online xxxx
Communicated by A. Fink

Keywords:
Integer partitions
Sand pile model
Garden of Eden states
Discrete dynamical systems
Enumeration

ABSTRACT

A Garden of Eden state in a dynamical system is one with no preimage. For operations on partitions known as the sand pile model and its generalizations, we characterize and enumerate these Garden of Eden partitions. In addition to motivation from the models themselves, there are also connections to the Rogers–Ramanujan identities and partitions with short sequences studied by Andrews.

1. Introduction

The sand pile model was introduced by Bak et al. [3] as a simple example in a theory of self-organized criticality in the setting of dynamical systems. Soon it was connected with earlier work of Brylawski [4] on the structure of integer partitions. A more complete summary of the history is provided after definitions are given. Our contributions here include characterizing which partitions have no preimages in the dynamical systems and various results on enumerating them.

2. Definitions

In this section we review the definitions we will use throughout the paper.

Definition 1. *A partition λ of a nonnegative integer n* is a finite nonincreasing sequence of positive integers $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ with $\sum \lambda_i = n$. We call n the size of λ, denoted $|\lambda|$. The λ_i are the *parts* of λ and k is the *number of parts* of λ, denoted $\#(\lambda)$. Also, it will sometimes be convenient to consider a partition having trailing zeros, but these are not counted as parts.

For the operations studied here, we represent a partition λ as k ordered columns of height λ_i thought of as piles of sand grains; see Fig. 1. In relation to more common representations, this is the 90° counterclockwise rotation of λ's Ferrers diagram, or equivalently, the conjugate of λ's "French" Ferrers diagram.

* Corresponding author.
E-mail address: bhopkins@saintpeters.edu (B. Hopkins).

http://dx.doi.org/10.1016/j.tcs.2017.06.024
0304-3975/© 2017 Elsevier B.V. All rights reserved.

$$\begin{array}{ccccc}
1 & \square & & & \\
1 & \square & & & \\
3 & \square & \square & \square & \\
4 & \square & \square & \square & \square \\
4 & \square & \square & \square & \square \\
 & 5 & 3 & 3 & 2
\end{array}$$

Fig. 1. The partition $\lambda = 5332$ has conjugate $\lambda' = 44311$.

Definition 2. The *conjugate* of a partition λ of n, denoted λ', is determined by the sequence λ'_i counting the number of parts of λ that are greater than or equal to i.

As with Ferrers diagrams, this is more easily seen as swapping the role of rows and columns; see Fig. 1.

Definition 3. Let $\mathcal{E}$ denote the set of partitions consisting of even parts and $\mathcal{O}$ the set of partitions consisting of odd parts.

A partition λ corresponds to an ordered pair $(\lambda_e, \lambda_o) \in \mathcal{E} \times \mathcal{O}$. E.g., $\lambda = 5332$ corresponds to $(2, 533)$. Note that $\#(\lambda) = \#(\lambda_e) + \#(\lambda_o)$.

Definition 4. Given a partition λ, let λ° be the partition obtained by removing all the parts of size 1 from λ.

Note that $\lambda = \lambda^\circ$ if and only if λ does not have 1 as a part. Also, unless λ consists only of 1s, the smallest part of λ° is at least 2.

Definition 5. If $\mathcal{S}$ is a set of partitions, let $p(\mathcal{S}, n)$ denote the number of partitions in $\mathcal{S}$ of size n and let $p(\mathcal{S}, m, n)$ denote the number of partitions in $\mathcal{S}$ of size n with m parts. The generating function for $\mathcal{S}$ is

$$F(\mathcal{S}, q) = \sum_{n=0}^{\infty} p(\mathcal{S}, n) q^n$$

and the multivariable generating function for $\mathcal{S}$ is

$$F(\mathcal{S}, x, q) = \sum_{n=0}^{\infty} p(\mathcal{S}, m, n) x^m q^n.$$

We use the standard notation for hypergeometric series (see [1]), e.g.,

$$(a; q)_n = (1 - a)(1 - aq)(1 - aq^2) \dots (1 - aq^{n-1}),$$
$$(a; q)_\infty = \lim_{n \to \infty} (a; q)_n.$$

2.1. Models

Definition 6. The *sand pile model* (SP) is the set of partitions together with the following transition rules.

(a) A grain of sand may fall from column i to column $i + 1$ (from the ith part to the $(i + 1)$st part) if $\lambda_i - \lambda_{i+1} \geq 2$. That is,

$$(\dots, \lambda_i, \lambda_{i+1}, \dots) \mapsto (\dots, \lambda_i - 1, \lambda_{i+1} + 1, \dots).$$

(b) A grain of sand may fall from the last column (smallest part) to create a new column of height 1 (part of size 1) if the last part $\lambda_k \geq 2$.

$$(\dots, \lambda_k) \mapsto (\dots, \lambda_k - 1, 1)$$

Fig. 2 shows applications of these transition rules. If we consider a partition to have trailing zeros, then (b) of this definition is redundant.

Since the sand pile model preserves partition size (i.e., it moves grains of sand but does not change the number of grains), we think of the model determining a dynamical system on the set of partitions of n. Fig. 3 shows the system determined by the sand pile model on the partitions of 6.

As with any finite dynamical system, two sets of partitions are distinguished by the operation, analogous to sinks and sources in a continuous system.

Please cite this article in press as: J. Anderson et al., Garden of Eden partitions in the sand pile and related models, Theoret. Comput. Sci. (2017), http://dx.doi.org/10.1016/j.tcs.2017.06.024