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The vulnerability of the diameter of the enhanced hypercubes

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ABSTRACT

Concern about fault tolerance in the design of interconnection networks has raised interest in the study of graphs such that deleting some vertices increases the diameter only moderately. For an interconnection network G, the $(\omega-1)$-*fault diameter* $D_\omega(G)$ is the maximum diameter of a subgraph obtained by deleting fewer than ω vertices of G, and the ω-*wide diameter* $d_\omega(G)$ is the least ℓ such that any two vertices are joined by ω internally-disjoint paths of length at most ℓ. The enhanced hypercube $Q_{n,k}$ is a variant of the well-known n-dimensional hypercube Q_n in which an edge is added from each vertex $x_n,\ldots,x_1$ to the vertex obtained by complementing $x_k,\ldots,x_1$. Yang, Chang, Pai, and Chan gave an upper bound for $d_{n+1}(Q_{n,k})$ and $D_{n+1}(Q_{n,k})$ and posed the problem of finding the wide diameter and fault diameter of $Q_{n,k}$. By constructing internally disjoint paths between any two vertices in the enhanced hypercube, for $n\ge 3$ and $2\le k\le n$ we prove that $D_\omega(Q_{n,k})=d_\omega(Q_{n,k})=d(Q_{n,k})$ for $1\le\omega<n-\lfloor\frac{k}{2}\rfloor$; $D_\omega(Q_{n,k})=d_\omega(Q_{n,k})=d(Q_{n,k})+1$ for $n-\lfloor\frac{k}{2}\rfloor\le\omega\le n+1$, where $d(Q_{n,k})$ is the diameter of $Q_{n,k}$. These results mean that interconnection networks modeled by enhanced hypercubes are extremely robust.

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1. Introduction

An interconnection network is conveniently represented by an undirected graph. The vertices (or edges) of the graph represent the nodes (or links) of the network. Throughout this paper, vertex and node, edge and link, graph and network are used interchangeably. Reliability and efficiency are important criteria in the design of interconnection networks. In the study of fault-tolerance and transmission delay of networks, wide diameter and fault diameter are important parameters that have been studied by many researchers. They combine connectivity with diameter to measure simultaneously the fault-tolerance and efficiency of parallel processing computer networks. These parameters were studied by several authors for some Cartesian product graphs [7,27,28] and for the hypercube and its variants [3,5,10,12,19–21].

Let u and v be two vertices in a network G. A u,v-*path* is a path with endpoints u and v. The *distance* between u and v, denoted by $d(u,v)$, is the minimum length (number of edges) of a u,v-path. The *diameter* of G, denoted by $d(G)$, is the maximum distance between vertices. The *connectivity* $\kappa(G)$ is the minimum number of vertices whose removal results in a disconnected or 1-vertex network. We say that G is ω-*connected* when $1\le\omega\le\kappa(G)$.

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Failures are inevitable when a network is put in use. Therefore, it is important to consider faulty networks. By the definition of the connectivity, if G is a ω-connected graph then removal of fewer than ω vertices from G still keeps the resulting graph connected. The $(\omega-1)$-*fault diameter* of a graph G, denoted by $D_\omega(G)$, is the maximum diameter among subgraphs obtained from G by deleting fewer than ω vertices; it measures the worst-case effect on the diameter when vertex faults occur. Note that $D_\omega(G)$ is well-defined if and only if G is ω-connected. Moreover,

$$d(G) = D_1(G) \le D_2(G) \le \cdots \le D_{k-1}(G) \le D_k(G).$$

The problem we are interested in is to determine the precise value of $D_\omega(G)$ or derive the relations between $D_\omega(G)$ and $d(G)$ for a given ω connected graph G. The answer to this problem is quite important for some applications in a real-time system. Generally speaking, this is a quite difficult problem. However, the precise values of $D_\omega(G)$ have been determined for some well-known networks G (see, for example, Section 13.6 in the text-book [26]).

By Menger's Theorem [17], in a ω-connected network there exist ω internally disjoint paths joining any two vertices (*internally disjoint* means that the only shared vertices are the endpoints). Given a ω-connected graph G, fix ω with $1 \le \omega \le k$. The *ω-wide diameter* of G, denoted by $d_\omega(G)$, is the least ℓ such that for any $u, v \in V(G)$ there exist ω internally disjoint u, v-paths of length at most ℓ. Throughout this paper, we abuse terminology by writing "disjoint paths" to mean "internally disjoint paths". Note that $d_1(G)$ is just the diameter $d(G)$ of G. From the definition,

$$d(G) = d_1(G) \le d_2(G) \le \cdots \le d_{k-1}(G) \le d_k(G).$$

An ideal interconnection network has connectivity as large as possible and diameter as small as possible. The wide diameter $d_\omega(G)$ combines connectivity $\kappa(G)$ with diameter $d(G)$, where $1 \le \omega \le \kappa(G)$. Hence $d_\omega(G)$ is a more suitable parameter than $\kappa(G)$ and $d(G)$ to measure fault-tolerance and efficiency of parallel processing computer networks. Determining the precise value of $d_\omega(G)$ is significant for a given graph and an integer ω with $1 \le \omega \le \kappa(G)$. However, Hsu [9] proved that this problem is NP-complete.

From the definitions, it follows that $D_\omega(G) \le d_\omega(G)$ when G is ω-connected. Equality holds for some well-known networks [6,14].

As a topology for an interconnection network of a multiprocessor system, the hypercube is a widely used and well-known model, since it possesses many attractive properties such as regularity, symmetry, logarithmic diameter, high connectivity, recursive construction, ease of bisection, and relatively low link complexity [13,18,25]. We study an important variant of the hypercube Q_n, the enhanced hypercube $Q_{n,k}$ proposed by Tzeng and Wei [22]; it has a smaller diameter, improved mean node distance and cost efficiency when compared to the hypercube; its properties have been studied in [2,15,23,24,29]. For $2 \le k \le n$, the *n-dimensional enhanced hypercube* $Q_{n,k}$ is obtained from the hypercube Q_n in which an edge is added from each vertex to the vertex obtained by complementing its last k bits. As a special case of the enhanced hypercube, the folded hypercube FQ_n was studied in [11,30,31]. We give the basic properties of $Q_{n,k}$ in Section 2.

It was shown by Liu [15] that $\kappa(Q_{n,k}) = n+1$. Thus, the wide diameter $d_\omega(Q_{n,k})$ and the fault diameter $D_\omega(Q_{n,k})$ are well-defined when $\omega \le n+1$. Yang, Chang, Pai, and Chan [29] gave an upper bound for $d_{n+1}(Q_{n,k})$ and $D_{n+1}(Q_{n,k})$, and they posed the problem of finding the wide diameter and fault diameter of $Q_{n,k}$. In this paper, for $n \ge 3$ and $2 \le k \le n$, we prove

$$D_\omega(Q_{n,k}) = d_\omega(Q_{n,k}) = \begin{cases} d(Q_{n,k}) & \text{for } 1 \le \omega < n - \lfloor \frac{k}{2} \rfloor; \\ d(Q_{n,k}) + 1 & \text{for } n - \lfloor \frac{k}{2} \rfloor \le \omega \le n+1. \end{cases}$$

The special case $k = n$ (folded hypercube) was obtain earlier by Simó and Yebra [20], along with the same values for edge deletions. For enhanced hypercubes also, our arguments yield the same values for edge deletions as for vertex deletions.

To prove this result, in Section 3 we explicitly construct disjoint paths of bounded length joining any two vertices in $Q_{n,k}$. The proofs of our main results are in Section 4.

2. Properties of $Q_{n,k}$

Let $x_n \cdots x_1$ be an n-bit binary string. We call the rightmost bit the first bit and the leftmost bit the nth bit. For simplicity we use a^i to mean that the bit a is repeated i times; for example, $01^3 0^2 = 011100$. The *Hamming distance* between strings u and v, denoted by $H(u, v)$, is the number of positions where the two strings differ.

The *n-dimensional hypercube* Q_n is the graph whose vertices are the n-bit binary strings and whose edges are the pairs of vertices differing in exactly one position. An edge of Q_n is a *j-dimensional edge* if the two endpoints differ in the jth position. For $1 \le j \le n$, let E_j denote the set of j-dimensional edges in Q_n.

As a variant of the hypercube, the *n-dimensional folded hypercube* FQ_n, proposed first by El-Amawy and Latifi [1], is obtained from the hypercube Q_n by making each vertex u adjacent to its complementary vertex, denoted $\bar{u}$ and obtained from u by subtracting each bit from 1. Such an edge is often called a *complementary edge*.

As we have defined, the enhanced hypercube $Q_{n,k}$ for $2 \le k \le n$ is obtained from the hypercube Q_n by adding the edge uv whenever u and v are related by $u = x_n \cdots x_1$ and $v = x_n \cdots x_{k+1}\bar{x}_k\bar{x}_{k-1} \cdots \bar{x}_1$, that is, the k bits $x_k, \cdots, x_1$ are complemented. Such an edge is called a *k-complementary edge*. For convenience, we use E_0 to denote the set of k-complementary edges. Thus $E(Q_{n,k}) = E(Q_n) \cup E_0$. When $k = n$, we have $Q_{n,n} = FQ_n$; hence the enhanced hypercube is a generalization

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