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On the minimum eccentricity shortest path problem ☆

Feodor F. Dragan *, Arne Leitert

Department of Computer Science, Kent State University, Kent, OH 44240, USA

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ABSTRACT

In this paper, we introduce and investigate the *Minimum Eccentricity Shortest Path (MESP)* problem in unweighted graphs. It asks for a given graph to find a shortest path with minimum eccentricity. Let n and m denote the number of vertices and the number of edges of a given graph. We demonstrate that:

- a minimum eccentricity shortest path plays a crucial role in obtaining the best to date approximation algorithm for a minimum distortion embedding of a graph into the line;
- the MESP problem is NP-hard for planar bipartite graphs with maximum degree 3 and W[2]-hard for general graphs;
- a shortest path of minimum eccentricity k can be computed in $\mathcal{O}(n^{2k+2}m)$ time;
- a 2-approximation, a 3-approximation, and an 8-approximation for the MESP problem can be computed in $\mathcal{O}(n^3)$ time, in $\mathcal{O}(nm)$ time, and in $\mathcal{O}(m)$ time, respectively;
- in a graph with a shortest path of eccentricity k, a k-dominating set can be found in $n^{\mathcal{O}(k)}$ time.

1. Introduction

All graphs occurring in this paper are connected, finite, unweighted, undirected, loopless and without multiple edges. For a graph $G = (V, E)$, we use $n = |V|$ and $m = |E|$ to denote the cardinality of the vertex set and the edge set of G. For a vertex v of G, $N_G(v) = \{u \in V \mid uv \in E\}$ is called the *open neighbourhood*, and $N_G[v] = N_G(v) \cup \{v\}$ the *closed neighbourhood* of v.

The *length* of a path from a vertex v to a vertex u is the number of edges in the path. The *distance* $d_G(u, v)$ of two vertices u and v is the length of a shortest path connecting u and v. The distance between a vertex v and a set $S \subseteq V$ is defined as $d_G(v, S) = \min_{u \in S} d_G(u, v)$. The *eccentricity* $\mathrm{ecc}_G(v)$ of a vertex v is $\max_{u \in V} d_G(u, v)$. For a set $S \subseteq V$, its eccentricity is $\mathrm{ecc}_G(S) = \max_{u \in V} d_G(u, S)$.

In this paper, we investigate the following problem.

Definition 1 *(Minimum eccentricity shortest path problem).* For a given graph G, find a shortest path P such that, for each shortest path Q, $\mathrm{ecc}_G(P) \leq \mathrm{ecc}_G(Q)$.

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* Corresponding author.
E-mail addresses: dragan@cs.kent.edu (F.F. Dragan), aleitert@cs.kent.edu (A. Leitert).

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Although this problem might be of an independent interest (it may arise in determining a "most accessible" speedy linear route in a network and can find applications in communication networks, transportation planning, water resource management and fluid transportation), our interest in this problem stems from the role it plays in obtaining the best to date approximation algorithm for a minimum distortion embedding of a graph into the line. In Section 2, we demonstrate that every graph G with a shortest path of eccentricity k admits an embedding f of G into the line with distortion at most $(8k+2)\,\mathrm{ld}(G)$, where $\mathrm{ld}(G)$ is the minimum line-distortion of G. Furthermore, if a shortest path of G of eccentricity k is given in advance, then such an embedding f can be found in linear time.

This fact augments the importance of investigating the *Minimum Eccentricity Shortest Path* problem (MESP problem) in graphs. Fast algorithms for it will imply fast approximation algorithms for the minimum line distortion problem. Existence of low eccentricity shortest paths in special graph classes will imply low approximation bounds for those classes. For example, all AT-free graphs (and hence all interval, permutation, cocomparability graphs) enjoy a shortest path of eccentricity at most 1 [4], all convex bipartite graphs enjoy a shortest path of eccentricity at most 2 [9].

We prove also that for every graph G with $\mathrm{ld}(G) = \lambda$, the minimum eccentricity of a shortest path of G is at most $\lfloor \frac{\lambda}{2} \rfloor$. Hence, one gets an efficient embedding of G into the line with distortion at most $\mathcal{O}(\lambda^2)$.

In Section 3, we show that the MESP problem is NP-hard for bipartite planar graphs with maximum degree 3, W[2]-hard on general graphs, and that a shortest path of minimum eccentricity k, in general graphs, can be computed in $\mathcal{O}(n^{2k+2}m)$ time. In Section 4, we design, for the MESP problem on general graphs, a 2-approximation algorithm that runs in $\mathcal{O}(n^3)$ time, a 3-approximation algorithm that runs in $\mathcal{O}(nm)$ time and an 8-approximation algorithm that runs in linear time. In Section 5, we will show that in a graph with a shortest path of eccentricity k a k-dominating set can be found in $n^{\mathcal{O}(k)}$ time.

Note that our Minimum Eccentricity Shortest Path problem is close but different from the *Central Path* problem in graphs introduced in [21]. It asks for a given graph G to find a path P (not necessarily shortest) such that any other path of G has eccentricity at least $ecc_G(P)$. The Central Path problem generalizes the Hamiltonian Path problem and therefore is NP-hard even for chordal graphs [20]. Our problem is polynomial time solvable for chordal graphs [10].

In what follows, we will need the following additional notions and notations.

The *diameter* of a graph G is $\mathrm{diam}(G) = \max_{u,v\in V} d_G(u,v)$. The diameter $\mathrm{diam}_G(S)$ of a set $S \subseteq V$ is defined as $\max_{u,v\in S} d_G(u,v)$. A pair of vertices x, y of G is called a *diametral pair* if $d_G(u,v) = \mathrm{diam}(G)$. In this case, every shortest path connecting x and y is called a *diametral path*.

A path P of a graph G is called a *k-dominating path* of G if $ecc_G(P) \le k$. In this case, we say also that P *k-dominates* each vertex of G. A pair of vertices x, y of G is called a *k-dominating pair* if every path connecting x and y has eccentricity at most k.

For a vertex s, let $L_i^{(s)} = \{\, v \mid d_G(s,v) = i \,\}$ denote the vertices with distance i from s. We will also refer to $L_i^{(s)}$ as the i-th layer.

2. Motivation through the line-distortion of a graph

Computing a minimum distortion embedding of a given n-vertex graph G into the line ℓ was recently identified as a fundamental algorithmic problem with important applications in various areas of computer science, like computer vision [22], as well as in computational chemistry and biology (see [16,17]). The *minimum line distortion* problem asks, for a given graph $G = (V, E)$, to find a mapping f of vertices V of G into points of ℓ with minimum number λ such that $d_G(x,y) \le |f(x) - f(y)| \le \lambda\, d_G(x,y)$ for every $x, y \in V$. The parameter λ is called the *minimum line-distortion* of G and denoted by $\mathrm{ld}(G)$. The embedding f is called *non-contractive* since $d_G(x,y) \le |f(x) - f(y)|$ for every $x, y \in V$.

In [2], Bădoiu et al. showed that this problem is hard to approximate within some constant factor. They gave an exponential-time exact algorithm and a polynomial-time $\mathcal{O}(n^{1/2})$-approximation algorithm for arbitrary unweighted input graphs, along with a polynomial-time $\mathcal{O}(n^{1/3})$-approximation algorithm for unweighted trees. In fact, their algorithms achieve line-distortion $O(\lambda^2)$ for general (unweighted) graphs, and line-distortion $O(\lambda^{3/2})$ for unweighted trees, where λ is the minimum line-distortion. In another paper [1], Bădoiu et al. showed that the problem is hard to approximate by a factor $\mathcal{O}(n^{1/12})$, even for weighted trees. They also gave a better polynomial-time approximation algorithm for general weighted graphs, along with a polynomial-time algorithm that approximates the minimum line-distortion λ embedding of a weighted tree by a factor that is polynomial in λ.

Fast exponential-time exact algorithms for computing the line-distortion of a graph were proposed in [5,12,13]. Fomin et al. [13] showed that a minimum distortion embedding of an unweighted graph into the line can be found in time $5^{n+o(n)}$. Fellows et al. [12] gave an $\mathcal{O}(n\lambda^4(2\lambda+1)^{2\lambda})$ time algorithm that for an unweighted graph G and integer λ either constructs an embedding of G into the line with distortion at most λ, or concludes that no such embedding exists. They extended their approach also to weighted graphs obtaining an $\mathcal{O}(n\lambda^{4W}(2\lambda+1)^{2\lambda W})$ time algorithm, where W is the largest edge weight. Thus, the problem of minimum distortion embedding of a given n-vertex graph G into the line ℓ is Fixed Parameter Tractable. Recently, Cygan and Pilipczuk [5] enhanced the $5^{n+o(n)}$ time and $\mathcal{O}^*(2^n)$ space algorithm by Fomin et al. [13] to an algorithm working in $\mathcal{O}(4.383^n)$ time and space.

Heggernes et al. [14,15] initiated the study of minimum distortion embeddings into the line of specific graph classes other than trees. In particular, they gave polynomial-time algorithms for the problem on bipartite permutation graphs and on threshold graphs [15]. Furthermore, in [14], Heggernes et al. showed that the problem of computing a minimum distortion embedding of a given graph into the line remains NP-hard even when the input graph is restricted to a bipartite,

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