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On cartesian closed extensions of non-pointed domains

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A B S T R A C T

Let **P** denote a cartesian closed full subcategory of the category **POSET** of posets and Scott continuous functions. We define **C-P** to be the full subcategory of objects from **P** whose D-completion is isomorphic to an object from **C**, where **C** is a subcategory of the category **CONT** of domains. The category **C-P** is always a subcategory of the category **CONTP** of continuous posets and Scott continuous functions. We prove that if **C** is a cartesian closed full subcategory of **F-L**, **U-L**, **F-RB** or **U-RB**, then the category **C-P** is also cartesian closed. It is known that the category **CDCPO** of consistent directed complete posets and Scott continuous functions is cartesian closed. In particular, we have the following cartesian closed categories: **F-L-CDCPO**, **U-L-CDCPO**, **F-RB-CDCPO**, **U-RB-CDCPO**, **F-aL-CDCPO**, **U-aL-CDCPO**, **F-B-CDCPO**, **U-B-CDCPO**, etc. If the categories **FS** and **RB** coincide, then it leads directly to the most potential of this uniform way of finding new cartesian closed categories of continuous posets: for every cartesian closed full subcategory **C** of **CONT**, the category **C-P** is also cartesian closed.

1. Introduction

It is well known that the category $\mathbf{CONT}_\perp$ of pointed domains and Scott continuous functions is not cartesian closed. Finding the maximal cartesian closed subcategories of $\mathbf{CONT}_\perp$ used to be a long-standing problem (we mean full subcategories whenever we talk about subcategories in this paper). In [7], Jung successfully classified all cartesian closed subcategories of $\mathbf{CONT}_\perp$: every cartesian closed subcategory of $\mathbf{CONT}_\perp$ is contained in **L**, the category of L-domains, or **FS**, the category of FS-domains. The cartesian closed category **RB** of retracts of bifinite domains is contained in **FS**. However, it is still not known whether **FS** and **RB** coincide. Passing from pointed domains to general domains which do not necessarily have a least element, it turns out that there are four maximal cartesian closed subcategories of the category **CONT** of domains: **F-L**, **F-FS**, **U-L** and **U-FS**, where the notation **F-C** (**U-C**) denotes the category whose objects are finite amalgams (disjoint unions) of objects from another category **C** (see Definition 4.3.7 in [1]). Correspondingly, there are another two cartesian closed categories: **F-RB** and **U-RB** (also denoted by **cUB** and **cFB** in [6]), and their maximality is still unknown.

One direction of the research of domain theory is to generalize the properties of continuous dcpos to the case of continuous posets (see e.g. [5], [8]). It is natural to look for cartesian closed categories consisting of continuous posets rather than domains. To allow the rich theory of cartesian closedness, that has been developed in the setting of domains, to be transformed to the more general setting with a minimum of effort, Zhang et al. [11] defined subcategories of **CONTP**, the category of continuous posets and Scott continuous functions, by using the D-completion (see [12]): let **C-P** denote the category of continuous posets whose D-completion is isomorphic to an object from a subcategory **C** of **CONT**. If the category **POSET** is

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cartesian closed, then the exponential objects must be isomorphic to the function spaces. And by the results proved in [11], we have that if **C** is a cartesian closed subcategory of $\mathbf{ALG}_\perp$, the category of pointed algebraic domains, or **BC**, the category of bounded complete domains, then the D-completion of the function space of any two objects of **C-P** is isomorphic to the function space of their D-completions, and thus **C-P** is cartesian closed. There exist two posets such that the evaluation map is not Scott continuous (see Example 2.1 in [3]). Thus the category **POSET** is not cartesian closed. However, we can turn to the setting of cartesian closed subcategories of **POSET**. For example, the category **CDCPO** of those posets, called cdcpos, in which every bounded directed subset has a supremum is cartesian closed (Corollary 2.1 in [3]). Notice that a poset is continuous (resp. pointed) if and only if its D-completion is a domain (resp. pointed dcpo) (see Theorems 1, 4 in [12]). Immediately, one has a series of cartesian closed subcategories of the category $\mathbf{CONTP}_\perp$ of pointed continuous posets, such as **BC-CDCPO**, **LAT-CDCPO**, **aL-CDCPO**, **B-CDCPO**, **aBC-CDCPO**, **aLAT-CDCPO**, etc., where **C-CDCPO** consists of cdcpos whose D-completion is isomorphic to an object of **C**. The main purpose of this paper is to further strengthen the results, especially, to extend the case of pointed domains to the case of non-pointed domains.

2. Preliminaries

We assume that the reader is familiar with the symbol system and definitions in [11], which we shall often refer to. We also use the standard definitions of domain theory as can be found in [1], [4] and [6].

We use **P** to denote a cartesian closed subcategory of the category **POSET** of posets and Scott continuous functions (or continuous functions for short). The morphisms of the categories considered in this paper are all continuous functions. It is easy to check that the one-point domain $T = \{*\}$ is a terminal object of **P** and the categorical product of two objects in **P** is isomorphic to their cartesian product. Let P and Q be two posets in **P**, and Q^P be an exponential object for P and Q with a morphism $ev : Q^P \times P \to Q$ such that for each $g : R \times P \to Q$ there exists a unique morphism $\Lambda_g : R \to Q^P$ such that $ev \circ (\Lambda_g \times id_P) = g$. Then there is a natural isomorphism $\alpha : [P \to Q] \to Q^P$ defined by $\alpha(f) = \Lambda_{f'}(*)$, where $f' \in [T \times P \to Q]$ such that $f'(*, p) = f(p)$ for all $p \in P$. Let $EV : [P \to Q] \times P \to Q$ be the evaluation function which is defined by $EV(f, p) = f(p)$. Then $EV = ev \circ (\alpha \times id_P)$ is continuous and hence $[P \to Q]$ together with EV is also an exponential object for P and Q.

Proposition 2.1. *Let P, Q be objects of* **P** *and let $\{f_i : i \in I\}$ be a directed subset of $[P \to Q]$. Consider the following conditions:*

(1) $\bigvee_{i\in I} f_i$ *exists;*

(2) $\bigvee_{i\in I} f_i(p)$ *exists for all $p \in P$;*

(3) $\forall p \in P$, $f(p) = \bigvee_{i\in I} f_i(p)$ *where* $f = \bigvee_{i\in I} f_i$.

Then (1) *and* (2) *are equivalent and both imply* (3).

Proof. The proof follows immediately from the above fact that the evaluation function $EV : [P \to Q] \times P \to Q$ is continuous. □

A D-completion $(D(P), \eta_P)$ of a poset P is a dcpo $D(P)$ together with a continuous function $\eta_P : P \to D(P)$, such that for any continuous function $f : P \to M$ into a dcpo M there exists a unique continuous function $\hat{f} : D(P) \to M$ satisfying $f = \hat{f} \circ \eta_P$. We take P as a subset of $D(P)$ with η_P as the inclusion map. There exists a unique order embedding $E : [P \to Q] \to [D(P) \to D(Q)]$ such that $\eta_Q \circ f = E(f) \circ \eta_P$ for all $f \in [P \to Q]$.

Lemma 2.2. (*See* [11].) (1) *For every $g \in [D(P) \to D(Q)]$ with $g(P) \subseteq Q$, the restriction $g|_P : P \to Q$ is continuous and $E(g|_P) = g$, hence $g \in E([P \to Q])$.*

(2) *If P and Q are objects of* **P***, then the function $E : [P \to Q] \to [D(P) \to D(Q)]$ is continuous.*

Let L be a domain and P a poset. A subset $B \subseteq L$ is called a basis of L if for all $x \in L$, $\downarrow x \bigcap B$ is a directed set with supremum x. If there is a continuous and order-embedding function $j : P \to L$ such that $j(P)$ is a basis of L, then (P, j) or P is called an embedded basis of L, and we take P as a subset of L with j as the inclusion function (see [9] for a more general case). The following theorem characterizes the D-completion by the concept of an embedded basis:

Theorem 2.3. (*See* [11].) *Let $f : P \to L$ be a function from a poset P to a domain L. Then (L, f) is a D-completion of P if and only if (P, f) is an embedded basis of L.*

Lemma 2.4. *Let P, Q be objects of* **P**. *If the function space $[D(P) \to D(Q)]$ is a domain and $E([P \to Q])$ is a basis of $[D(P) \to D(Q)]$, then $D([P \to Q]) \cong [D(P) \to D(Q)]$.*

Proof. By Lemma 2.2(2), we have that the function $E : [P \to Q] \to [D(P) \to D(Q)]$ is a continuous embedding. Then $[P \to Q]$ is an embedded basis of $[D(P) \to D(Q)]$ and thus $D([P \to Q]) \cong [D(P) \to D(Q)]$ by Theorem 2.3. □