Doctopic: Logic, semantics and theory of programming ARTICLE IN PRESS TCS:11256

Theoretical Computer Science ••• (••••) •••–•••

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

A domain-theoretic approach to Brownian motion and general continuous stochastic processes

Paul Bilokon *, Abbas Edalat

Department of Computing, Imperial College London, United Kingdom

A R T I C L E I N F O

Article history:
Received 27 January 2016
Received in revised form 2 June 2017
Accepted 26 July 2017
Available online xxxx
Communicated by A. Jung

Keywords:
Brownian motion
Wiener measure
Domain theory
Stochastic processes
Computability

A B S T R A C T

We introduce a domain-theoretic framework for continuous-time, continuous-state stochastic processes. The laws of stochastic processes are embedded into the space of maximal elements of the normalised probabilistic power domain on the space of continuous interval-valued functions endowed with the relative Scott topology. We use the resulting ω-continuous bounded complete dcpo to obtain partially defined stochastic processes and characterise their computability. For a given continuous stochastic process, we show how its domain-theoretic, i.e., finitary, approximations can be constructed, whose least upper bound is the law of the stochastic process. As a main result, we apply our methodology to Brownian motion. We construct a partially defined Wiener measure and show that the Wiener measure is computable within the domain-theoretic framework.

1. Introduction

This work is motivated by a desire to improve our understanding of stochastic processes, particularly in the light of recursion theory. In recent decades, major advances in stochastic calculus have been motivated by applications in the rapidly expanding field of quantitative finance. Stochastic processes have many other important applications, notably in filtering problems, stochastic approaches to deterministic boundary value problems, optimal stopping, and stochastic control [29].

Several schools of computable analysis have addressed the subject of measure theory and integration. The computability of measures and integration on the unit interval was studied in the light of Type-2 Theory of Effectivity (TTE) [34]. The computability of measures and set-theoretical operations was examined in the setting of a computable measure space [37, 38]. Computable probability frameworks were used to study Martin–Löf and dynamical randomness [17]. In a separate strand of literature, interval-valued and fuzzy-valued random variables have been considered, and there have also been extensions of stochastic integration to interval-valued and set-valued processes [39,26].

This article follows the tradition of applying domain theory [31] to classical analysis, which started with applications to dynamical systems, measures and fractals [7] and integration [5]. In this approach the classical spaces are realised as a subset of maximal elements of an ω-continuous dcpo, where the set of maximal elements is endowed with the relative Scott topology. By embedding the set of probability measures of any locally compact second countable metric space into the set of maximal elements of the probabilistic power domain [18] of the upper space of the metric space [7] a new theory of approximation of measures was obtained. This resulted in a generalisation of the Riemann integral to the so-called

* Corresponding author.
E-mail addresses: paul@bilokon.co.uk (P. Bilokon), a.edalat@imperial.ac.uk (A. Edalat).

http://dx.doi.org/10.1016/j.tcs.2017.07.016
0304-3975/© 2017 Published by Elsevier B.V.

Please cite this article in press as: P. Bilokon, A. Edalat, A domain-theoretic approach to Brownian motion and general continuous stochastic processes, Theoret. Comput. Sci. (2017), http://dx.doi.org/10.1016/j.tcs.2017.07.016

R-integral [5]. When the embedding is onto the set of maximal elements of such a dcpo, then the classical space is precisely a complete metrisable separable metric space [22].

More generally, as in the context of the present paper, when a separable metric space is homeomorphic to a G_δ subset of the maximal elements of an ω-continuous dcpo endowed with the Scott topology, the space of probability measures of the metric space endowed with the weak topology is homeomorphic to a subset of the maximal elements of the probabilistic power domain of the ω-continuous dcpo [9]. This result establishes a connection between the classical measure theory and domain theory with applications in various areas [8].

In [6], domain theory has been applied to discrete time stochastic processes. Here we follow a different path and develop a more general approach. We consider continuous time, continuous space stochastic processes through the prism of domain theory. Not only does this theoretical apparatus allow us to examine the question of computability, it naturally yields new approaches to computation of stochastic processes by constructing a new data type for representing them.

The plan for this paper is as follows. In Sec. 2 we present some domain-theoretic and topological preliminaries. In Sec. 3, we introduce a domain-theoretic framework for discrete-time stochastic-processes. In the next section, Sec. 4, we use this framework as an inspiration, while replacing the domain-theoretic product spaces with the domain-theoretic function spaces to cater for the case of continuous-time, continuous-state stochastic processes. We consider the stochastic processes with the underlying compact-open topology of the space of trajectories and embed them into the set of maximal elements of the normalised probabilistic power domain of the space of Scott continuous interval-valued functions or trajectories, which extend the classical notion of trajectories of stochastic processes in the domain-theoretic setting. We derive a necessary and sufficient condition for the least upper bound of an increasing sequence of simple valuations in the normalised probabilistic power domain that, in effect, converges to the law of a stochastic process.

For a given classical Borel measure, we construct an increasing sequence of simple valuations on the normalised probabilistic power domain of the bounded complete ω-continuous dcpo whose least upper bound is the given measure. In particular, for a given continuous stochastic process, this yields a domain-theoretic approximation by partially defined stochastic processes (Sec. 5). We then formulate a notion of computability for partially defined stochastic processes which is used to define domain-theoretic computability for a classical stochastic process (Sec. 6).

As one of our main results, we apply our methodology to Brownian motion and its law, the Wiener measure (Sec. 7). Brownian motion is the stochastic process W defined by the following three properties: (i) $W_0 = 0$, (ii) the function $t \to W_t$ is almost surely everywhere continuous, (iii) W has independent increments with $W_t - W_s$ normally distributed with expected value 0 and variance $t - s$. The Wiener measure of a basic point-open set of continuous functions from $[0, 1]$ to $\mathbb{R}$, i.e. a set of the form

$$\{f \mid a_i < f(t_i) < b_i, 0 = t_0 < t_1 < \ldots, < t_n = 1\},$$

is given by

$$\frac{1}{\sqrt{\pi^n \prod_{i=1}^{n}(t_i - t_{i-1})}} \int_{a_1}^{b_1} \ldots \int_{a_n}^{b_n} \exp\left(\sum_{j=1}^{n} \frac{(x_j - x_{j-1})^2}{t_j - t_{j-1}}\right) dx_n \ldots dx_1, \tag{1}$$

where $x_0 := 0$. The computability of the Wiener measure and Brownian motion has previously been studied by Fouché [12, 4]. In particular, when all a_i, b_i and t_i are computable real numbers, the real number given by (1) is also computable.

The Lebesgue-type integral of a functional with respect to the Wiener measure is known as the Wiener integral whose computation presents significant challenges even for the simplest functionals [33]; research has focussed on the computation of several special cases [20,3]. Wiener measure and integration play a major rôle in stochastic analysis due to their association with Brownian motion and have found major applications in quantum physics [32] such as Feynman integration.

Due to the central role played by the Wiener measure in stochastic analysis and theoretical physics, the question of its computability attracted the attention of researchers working in the field of computable analysis. This question was addressed by Fouché. In [12], he considered ways in which the Brownian motion can be approximated by oscillations which are encoded by finite binary strings of high descriptive complexity. This enabled him to deduce recursive properties of this stochastic process. In [13], he showed that Brownian motion can be computed from an infinite binary string which is complex in the sense of Kolmogorov–Chaitin. He presents a direct construction of complex oscillations from algorithmically random real numbers based on ideas underlying the construction of Gaussian processes as spirals in Gaussian Hilbert spaces. This is similar to Wiener's Fourier analytic approach in that the Brownian motion is regarded as a random signal with random, normally distributed amplitudes. An investigation of the computability of this construction is presented in [4].

In this paper we develop a domain-theoretic method for approximating stochastic processes. We then show that the Wiener measure is domain-theoretically computable by using, among other things, a result by Paul Lévy [25]. This provides an alternative proof of the computability of the compact-open sets to that discovered by Fouché [12] and enables us to create a domain-theoretic approximation for Brownian motion.

The application of the domain-theoretic machinery to this problem creates many possibilities for further work, some of which are listed in Sec. 8.

Please cite this article in press as: P. Bilokon, A. Edalat, A domain-theoretic approach to Brownian motion and general continuous stochastic processes, Theoret. Comput. Sci. (2017), http://dx.doi.org/10.1016/j.tcs.2017.07.016