Theoretical Computer Science 693 (2017) 1–12

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

# The undecidability of arbitrary arrow update logic

Hans van Ditmarsch [a], Wiebe van der Hoek [b], Louwe B. Kuijer [b,a,*]

[a] LORIA, Campus scientifique, BP 239, 54506 Vandoeuvre-lès-Nancy Cedex, France
[b] Department of Computer Science, University of Liverpool, Ashton Building, Ashton Street, Liverpool L69 3BX, United Kingdom

ARTICLE INFO

*Article history:*
Received 10 March 2017
Received in revised form 7 July 2017
Accepted 26 July 2017
Available online 4 August 2017
Communicated by D. Sannella

*Keywords:*
Modal logic
Dynamic epistemic logic
Update logics
Undecidability
Satisfiability

ABSTRACT

Arbitrary Arrow Update Logic is a dynamic modal logic with a modality to quantify over arrow updates. Some properties of this logic have already been established, but until now it remained an open question whether the logic's satisfiability problem is decidable. Here, we show by a reduction of the tiling problem that the satisfiability problem of Arbitrary Arrow Update Logic is co-RE hard, and therefore undecidable.



## 1. Introduction

*Update Logics* are logics that provide an object language in which one can reason about the effect of changes to a model for that language. Such an underlying model is usually a Kripke model, equipped with a set of states and some relations between them. One of the most prominent examples of updates relate to the incorporation of new *information*. This field of studies has become popular as *Dynamic Epistemic Logic (DEL)* [7] in the past decades. In epistemic logic, states in a Kripke model represent a description of the world, and the relations represent 'possibility' (for belief) or 'indistinguishability' (for knowledge) relations. We say that $\Box\varphi$ is true in state $s$ in model $\mathcal{M}$, written $\mathcal{M}, s \models \Box\varphi$, if for all $t$, if $(s,t) \in R(a)$ then $\mathcal{M}, t \models \varphi$; that is, if in all states that are indistinguishable for agent $a$, formula $\varphi$ holds.

Keeping this epistemic setting in mind for the moment, *Public Announcement Logic (PAL)* [12,4], studies updates in which certain states of $\mathcal{M}$ are removed: $[\varphi]\psi$ means that after the announcement $\varphi$ (which is interpreted as the operation in which only the $\varphi$-states are retained in the model), $\psi$ holds. For example, if $\varphi$ means "the door is locked" and $\psi$ means "agent $a$ believes she cannot access the room", then $[\varphi]\psi$ means "after it is announced that the door is locked, agent $a$ will believe that she cannot access the room."

In *Arrow Update Logic (AUL)* [11], updates take the form of removing some *access* between states: $[(\varphi, a, \chi)]\psi$ denotes that if we only keep connections between two states if they are labeled $a$ and go from a $\varphi$ state to a $\chi$ state, $\psi$ will hold. For example, for the same meaning of $\varphi$ and $\psi$ as above, $[(\varphi, a, \varphi)]\psi$ means "if whenever the door is locked ($\varphi$) agent $a$ is told so ($\varphi = \chi$), then she (correctly) believes that she cannot access the room ($\psi$)".

* Corresponding author at: Department of Computer Science, University of Liverpool, Ashton Building, Ashton Street, Liverpool L69 3BX, United Kingdom.
*E-mail addresses:* hans.van-ditmarsch@loria.fr (H. van Ditmarsch), wiebe.van-der-hoek@liverpool.ac.uk (W. van der Hoek), Louwe.Kuijer@liverpool.ac.uk (L.B. Kuijer).

http://dx.doi.org/10.1016/j.tcs.2017.07.018
0304-3975/

Arrow updates are more powerful than public announcements; unlike public announcements, arrow updates can be used to model situations where different agents gain different information. For example, $a$ might be told whether the door is locked while $b$ is left in the dark on the matter. However, arrow updates can only *remove* arrows, they cannot *add* them. As a result, arrow updates can only be used to model situations where the amount of uncertainty decreases. If we want to model situations where the amount of uncertainty increases we will need to use an even more powerful kind of update. Among these more powerful kinds of updates, the most commonly used are *action models* [4]. Action models can, for example, be used to model the event where, from agent $b$'s perspective, it is possible that $a$ is told about whether the door is locked but it is also possible that $a$ is not told.

The logics using public announcements, arrow updates and action models are called Public Announcement Logic (*PAL*), Arrow Update Logic (*AUL*) and Action Model Logic (*AML*),[1] respectively.

For each of these logics there is also an "arbitrary" version: for *PAL* there is Arbitrary Public Announcement Logic (*APAL*) [3], for *AUL* there is Arbitrary Arrow Update Logic (*AAUL*) [8] and for *AML* there is Arbitrary Action Model Logic (*AAML*) [10]. These "arbitrary" logics contain an operator that quantifies over their non-arbitrary counterpart. So in *APAL* we have $[!]\psi$ if and only if $[\varphi]\psi$ holds for every *PAL* formula $\varphi$, in *AAUL* we have $[\updownarrow]\psi$ if and only if $[U]\psi$ for every *AUL* update $U$ and in *AAML* we have $[\times]\psi$ if and only if $[M]\psi$ for every *AML* action model $M$.

The logics *PAL*, *AUL*, and *AML* are equally expressive [4,11]. The arbitrary versions of the logics are not equally expressive, however. Under reasonable assumptions about the number of agents, the logics *APAL* and *AAUL* are incomparable in expressivity [8], and they are both strictly more expressive than *AAML* [3,8], since the latter logic is no more expressive than basic modal logic [10].

Two other logics that are similar to these "arbitrary" logics are Group Announcement Logic (*GAL*) [1] which allows quantification over a specific type of public announcements that are made by a group of agents, and Coalition Announcement Logic (*CAL*) [2] which allows us to ask whether there is some announcement for a group $G$ such that $\psi$ becomes true regardless of what all agents outside of $G$ announce.

It is important to realize that the relevance of this kind of updates goes beyond the realm of epistemic interpretations. In normative reasoning for instance, eliminating (bad) states enables one to reason about deontically 'better' situations, and eliminating (bad) transitions enforces 'better' behavior. For more on the epistemic and normative interpretations of updates, see [8, Section 2].

In this paper, we focus on *AAUL*. So we consider the operator $[\updownarrow]$ that quantifies over all arrow updates.

Several technical results regarding *AAUL* were established in [8]. Specifically, the following results were proven. *Expressivity*: [8] shows that, under some mild assumptions, *APAL* and *AAUL* are incomparable over the class of all Kripke models. A case in which *AAUL* is more expressive than *APAL* is also identified. Successively, *AAUL* is compared to a number of other logics: it is established that *AAUL* is incomparable to epistemic logic with common knowledge, but more expressive than *PAL*. It is known that basic epistemic logic, public announcement logic *PAL*, arbitrary action model logic *AAML*, and refinement modal logic [6] are all equally expressive. As a corollary of this result we therefore also have that *AAUL* is more expressive than *AAML*. *Model Checking*: [8] shows that the model checking problem for *AAUL* is PSPACE-complete. *Axiomatization*: An (infinitary) proof system for *AAUL* is introduced in [8] and its soundness and correctness (with respect to the set of intended models) is proven.

The question we address for *AAUL* in this paper regards its *decidability*. For some of the 'arbitrary' logics mentioned above, namely *APAL*, *GAL*, and *CAL*, the satisfiability problem is undecidable [9,2]. The satisfiability problem of *AAML*, on the other hand, is decidable [10]. For *AAUL*, it remained unknown whether the satisfiability problem is decidable. Here, we show that it is *not* decidable, by demonstrating that *AAUL*'s satisfiability problem can encode the tiling problem [14]. Because the tiling problem is known to be co-RE complete [5], this shows that the satisfiability problem of *AAUL* is co-RE hard.

The undecidability result is not surprising, but also not obvious. In *APAL*, *GAL*, and *CAL* the undecidability seems to originate in the semantic restriction of quantification: the quantification is *only* over quantifier-free formulas, not over all formulas; the resulting gaps in the quantification make these logics more expressive than epistemic logic, and this also seems to affect decidability. However, in *AAML* it does not matter if we so restrict the semantics of quantifiers: either way, we can eliminate quantifiers from the language by rewriting procedures, and epistemic logic is decidable. As *AAUL* seems half-way between *APAL* and *AAML*, the scales could have tilted both towards decidability and undecidability.

The undecidability proof presented here is similar to those in [9] and [2] in that they all use the "arbitrary" operators to encode a grid and then reduce the tiling problem to a satisfiability problem on that grid. The similarities between the proofs do not go far beyond that, however.

The structure of this paper is as follows. First, in Section 2 we introduce the syntax and semantics of *AAUL*. Then, in Section 3 we provide a brief definition of the tiling problem and show that it can be encoded in the satisfiability problem of *AAUL*.

## 2. AAUL syntax and semantics

Let $\mathcal{P}$ be a countable set of propositional variables and $\mathcal{A}$ a finite set of agents. We assume that $|\mathcal{A}| \geq 6$.

[1] *AML* is also sometimes referred to as Dynamic Epistemic Logic (*DEL*), but here we reserve that name for the family of update logics of which *AML* is one.