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Arc fault tolerance of Kautz digraphs

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ABSTRACT

The Kautz graphs are a famous class of network models and the super-λ property of graphs is a good indicator of the fault tolerance of networks. We, in this paper, study mainly the conditional arc connectivity and super-λ tolerance to arc-faults of digraphs, which are two measurements of super-λ property. We first give some bounds on them in terms of minimum degree of digraphs, and then determine them for Kautz digraphs. The obtained results show that the fault tolerance of Kautz networks is nice.

1. Terminology and introduction

1.1. Introduction

When a processor interconnection network is modeled by an undirected graph or a directed graph, its fault tolerance can be measured by the edge (arc) connectivity $\lambda = \lambda(D)$ of the corresponding undirected (directed) graph D. Clearly, the larger λ, the more fault-tolerant the corresponding network. Furthermore, if two graphs have the same edge (arc) connectivity λ, then the one with smaller m_λ is more fault-tolerant, where m_λ is the number of edge (arc) cuts of size λ. To maximize λ and minimize m_λ, super-λ undirected (directed) graphs were introduced by Bauer et al. [2] in 1981. Since then investigations on them have been made by many authors [5,11,13].

Two parameters have been introduced to measure the super-λ property of undirected graphs quantitatively. One is the restricted edge connectivity of undirected graphs, which was initially introduced by Esfahanian and Hakimi [4] and received much attention (cf. e.g. [1,10,12,15–17,19,21]). In Section 2, we first introduce the notion of conditional arc connectivity as an extension of restricted edge connectivity to directed graphs, and then give a sharp upper bound on conditional arc connectivity.

The other parameter for measuring super-λ property is the super-λ tolerance to edge-faults, which was first introduced by Hong et al. [6] and subsequently studied by several authors [7,18]. By replacing edges by arcs, one may introduce the concept of super-λ tolerance to arc-faults. However, little attention has been paid to this kind of fault tolerance. In Section 3, by means of the conditional arc connectivity, we show a characterization of super-λ directed graphs and present a bound on the super-λ tolerance to arc-faults.

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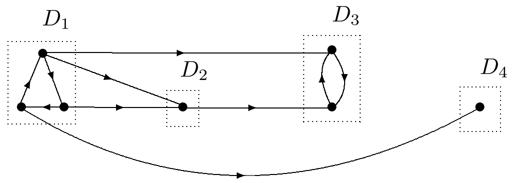

Fig. 1. A digraph with a TOSC D_1, D_2, D_3, D_4.

Due to their desirable properties, Kautz networks are viewed as strong competitors against the popular hypercube networks. Kautz graphs are the topology structures of Kautz networks, which have received much attention [5,9,12,15–17]. In particular, the restricted edge connectivity of undirected Kautz graphs and the super-λ property of directed Kautz graphs have been studied [5,12,15–17]. As applications of our results, we determine the conditional arc connectivity and the super-λ tolerance to arc-faults of directed Kautz graphs in Section 4.

1.2. Terminology and notations

By convention, undirected graphs and directed graphs are called graphs and digraphs, respectively. All digraphs considered here are finite and have neither loops nor parallel arcs. The *order* of D is the size $|V(D)|$ of its vertex set $V(D)$. A digraph is *trivial* if its order is 1; otherwise, it is *non-trivial*.

A (di)graph D is *connected* (*strong*) if there exists an undirected (directed) walk from every vertex to every other vertex in D. A trivial digraph is defined to be strong. For a connected graph (strong digraph) D, a set of edges (arcs) S of D is an *edge (arc) cut* if $D - S$ isn't connected (strong). The minimum size of an edge (arc) cut of a (di)graph D is called the *edge (arc) connectivity* of D and is denoted by $\lambda(D)$.

A digraph D is *connected* if the graph obtained from D ignoring the orientation of the arcs is connected. A maximal connected (strong) subdigraph of a digraph D is called a *connected (strong) component* of D. The strong components of D can be ordered as $D_1, \dots, D_t$ so that every arc between different strong components must leave the strong component with smaller subscript. Such an ordering is *a topological ordering of the strong components of D* or *a TOSC of D* for short. Fig. 1 shows a TOSC of a digraph.

For a digraph D and a pair Z, Z' of nonempty vertex subsets of D, the set of arcs from Z to Z', the set of arcs leaving Z, and the set of arcs entering Z are denoted by (Z, Z'), $\partial^+(Z)$, and $\partial^-(Z)$, respectively. We use subscripts (e.g. $\partial_D^-(Z)$) to specify the digraph D if necessary. We will not distinguish a set of cardinality 1 from its only element. For example, the arc set $\partial^-(\{u\})$ will be abbreviated to $\partial^-(u)$. For a vertex u of D, the cardinality of $\partial^-(u)$ is called the *in-degree* $d^-(u)$ of u and $\delta^-(D) = \min\{d^-(u) : u \in V(D)\}$ is called the *minimum in-degree* of D. The *out-degree* $d^+(u)$ of u and *minimum out-degree* $\delta^+(D)$ of D are defined analogously. The *minimum degree* $\delta(D)$ of D is $\min\{\delta^+(D), \delta^-(D)\}$.

2. Conditional arc connectivity of digraphs

2.1. Properties of conditional arc connectivity

We first introduce the concept of conditional arc connectivity.

Definition 2.1. Let D be a strong digraph and k a nonnegative integer. An arc cut S of D is *k-conditional* if $\delta(D - S) \geq k$. A k-conditional arc cut with minimum size is called a *$\lambda^{(k)}$-cut*. The size of a $\lambda^{(k)}$-cut of D is called the *k-conditional arc connectivity* of D and is denoted by $\lambda^{(k)}(D)$. A strong digraph containing $\lambda^{(k)}$-cuts is called *$\lambda^{(k)}$-connected*.

Clearly, the 0-conditional arc connectivity is exactly the arc connectivity. We mainly discuss the 1-conditional arc connectivity. For simplicity, the 1-conditional arc connectivity is called the *conditional arc connectivity*.

Next, we give the following simple but useful property of a $\lambda^{(1)}$-cut.

Theorem 2.1. *Let S be a $\lambda^{(1)}$-cut of a $\lambda^{(1)}$-connected digraph D and let $D_1, D_2, \dots, D_t$ be a TOSC of $D - S$. Then $|V(D_t)| \geq 2$, $|V(D_1)| \geq 2$, and $S = \partial^+(V(D_t)) = \partial^-(V(D_1))$.*

Proof. If $V(D_t) = \{v\}$, see Fig. 1 for an example. Clearly, $d^+_{D-S}(v) = 0$, contradicting the hypothesis that S is a conditional arc cut. Hence, $|V(D_t)| \geq 2$. Similarly, we have that $|V(D_1)| \geq 2$. By the definition of a TOSC, $\partial^+(V(D_t)) \subseteq S$. Clearly, $D - \partial^+(V(D_t))$ isn't strong and $\delta(D - \partial^+(V(D_t))) \geq \delta(D - S) \geq 1$. So $\partial^+(V(D_t))$ is also a conditional arc cut of D. By the minimality of S, we have $S = \partial^+(V(D_t))$. A similar argument shows $S = \partial^-(V(D_1))$. The proof is complete. □