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The entire chromatic number of graphs embedded on the torus with large maximum degree

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ABSTRACT

An embedded graph $G = (V, E, F)$ on the torus is entirely k-colorable if $V \cup E \cup F$ can be colored with k colors such that any two adjacent or incident elements receive different colors. In this paper, we prove that every embedded graph G on the torus with maximum degree $\Delta \geq 10$ is entirely $(\Delta + 2)$-colorable.

1. Introduction

Suppose that G is a simple graph. An embedding of G on a surface S is called a 2-*cell embedding* if each face of G is homeomorphic to an open unit disc. All embedding graphs considered in this paper are 2-cell embedding. The *Euler characteristic* $\varepsilon(S)$ of a surface S is equal to $V(G) + F(G) - E(G)$ for any graph G that is 2-cell embedded in S. If S is the Euclidean plane, then $\varepsilon(S) = 2$; If S is the torus, then $\varepsilon(S) = 0$. Given an embedded graph G, we denote its vertex set, edge set, face set, maximum degree, and minimum degree by $V(G), E(G), F(G), \Delta(G)$ and $\delta(G)$, respectively. If no ambiguity arises, $\Delta(G)$ is written as Δ. For convenience, a graph embedded on the torus is called a *T-graph*.

An *entire k-coloring* of an embedded graph G in a surface is a mapping $\phi : V(G) \cup E(G) \cup F(G) \to \{1, 2, \dots, k\}$ such that any two adjacent or incident elements in $V(G) \cup E(G) \cup F(G)$ receive distinct colors. The *entire chromatic number*, denoted $\chi_{vef}(G)$, of G is the smallest integer k such that G has an entire k-coloring.

In 1972, Kronk and Mitchem [4] proved that every plane graph G with $\Delta \leq 3$ is entirely $(\Delta + 4)$-colorable, and conjectured that $\chi_{vef}(G) \leq \Delta + 4$ for any plane graph G with $\Delta \geq 4$. The upper bound $\Delta + 4$ is tight since the complete graph K_4 satisfies $\chi_{vef}(K_4) = 7 = \Delta(K_4) + 4$. This conjecture has been solved completely (it was proved in [2] for $\Delta \geq 7$, in [6] for $\Delta = 6$, in [7] for $\Delta = 4, 5$). For the class of plane graphs of large maximum degree, the upper bound $\Delta + 4$ can be further improved. Wang, Mao and Miao [9] proved that every plane graph G with $\Delta \geq 8$ is entirely $(\Delta + 3)$-colorable. It is now known that if G is a plane graph with $\Delta \geq 9$, then its entire chromatic number is at most $\Delta + 2$ (this was

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proved in [1] for $\Delta \geq 12$ and in [8] for $9 \leq \Delta \leq 11$). Note that the upper bound $\Delta + 2$ cannot be further reduced for the class of plane graphs since any tree T with $\Delta \geq 2$ can attain this value. However, it is unknown what is the tight upper bound of $\chi_{vef}(G)$ for plane graphs G with $4 \leq \Delta \leq 8$. An easy observation is that for a wheel W_5 of five vertices, we have $\chi_{vef}(W_5) = 7 = \Delta(W_5) + 3$.

Sanders and Maharry [5] investigated the simultaneous colorings of embedded graphs. Among other things, they showed that if G is a T-graph with $\Delta \geq 51$, then $\chi_{vef}(G) \leq \Delta + 2$. Recently, the present four authors proved in [3] that if G is a T-graph, then $\chi_{vef}(G) \leq \Delta + 4$ if $\Delta \geq 6$, and $\chi_{vef}(G) \leq \Delta + 5$ if $\Delta \leq 5$. The conjecture that every T-graph G is entirely $(\Delta + 4)$-colorable, raised in [3], remains open.

In this paper, we will prove the following result:

Theorem 1. *If G is a T-graph with $\Delta \geq 10$, then $\chi_{vef}(G) \leq \Delta + 2$.*

Theorem 1 extends partially the result on the entire coloring of plane graphs in [8], also improves a result in [5] by reducing the value for Δ from 51 to 10.

2. Notations

Let G be a T-graph with $\delta(G) \geq 2$. For a face $f \in F(G)$, we use $b(f)$ to denote the boundary walk of f and write $f = [u_1u_2 \cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are the vertices of $b(f)$ in the clockwise order. Repeated occurrences of a vertex are allowed. The degree of a face, denoted $d(f)$, is the number of edge-steps in its boundary walk. Note that each cut-edge is counted twice. For $x \in V(G)$, let $d(x)$ denote the degree of x in G. A vertex of degree k (at most k, at least k, respectively) is called a k-vertex (k^--vertex, k^+-vertex, respectively). Similarly, we can define k-face, k^--face and k^+-face. For a vertex $v \in V(G)$, let $N(v)$ denote the set of neighbors of v in G. When v is a k-vertex, we say that there are k faces incident to v. However, these faces are not required to be distinct, i.e., v may have repeated occurrences on the boundary walk of some of its incident faces. We say that v is a $(a_1, a_2, \ldots, a_k)$-*vertex* if it is incident to k distinct faces $f_1, f_2, \ldots, f_k$ in the clockwise order with $d(f_i) = a_i$ for $i = 1, 2, \ldots, k$. For $x \in V(G) \cup F(G)$ and $i \geq 1$, let $n_i(x)$ (or $m_i(x)$) denote the number of i-vertices (or i-faces) adjacent or incident to x.

A vertex v is *weak* if $d(v) = 4$ and $m_3(v) \geq 1$, or if $d(v) = 5$ and $m_3(v) \geq 4$. A 4-face f is *weak* if $n_2(f) + n_3(f) + m_3(f) \geq 1$. A 5^+-face f is *weak* if $2n_2(f) + n_3(f) + m_3(f) + m_4^w(f) \geq 3d(f) - 11$, where $m_4^w(f)$ is the number of weak 4-faces adjacent to f. A 2-vertex is *bad* if it is incident to a 4-face, and *good* otherwise. Let $n_2^b(f)$ denote the number of bad 2-vertices incident to face f.

For an edge $e = xy \in E(G)$, let $t(e)$ denote the number of 3-faces incident to e; $q(e)$ denote the total number of 3-faces and weak 4^+-faces incident to e; and $p(e)$ denote the total number of 3^--vertices, weak 4-vertices and weak 5-vertices incident to e. Note that $p(e) \leq 2$ and $t(e) \leq q(e) \leq 2$. If $p(e) \geq 1$, $q(e) \geq 1$ and $d(x) + d(y) - p(e) - q(e) \leq \Delta - 1$, then e is called a *light edge*.

Given a face $f \in F(G)$, let $E^*(f) = \{xy \in b(f) \mid d(x) + d(y) \leq \Delta \text{ and } \min\{d(x), d(y)\} \leq 3\}$, and $\rho^*(f) = 3d(f) - 2n_2(f) - n_3(f) - m_3(f) - m_4^w(f) - |E^*(f)|$. If $|E^*(f)| \geq 1$ and $\rho^*(f) \leq 11$, then f is called a *light face*.

3. A structural lemma

This section is devoted to establish the following structural lemma, which is fundamentally applied to the proof of Theorem 1 in the next section.

Lemma 1. *Let G be a connected T-graph with $\Delta \geq 10$ and $\delta(G) \geq 2$. Then G contains one of the following configurations* (C6) *to* (C6)*:*

(C1) *a 2-vertex adjacent to two other 2-vertices;*
(C2) *a 2-vertex lying on a 3-face;*
(C3) *a 2-vertex lying on two weak faces, one of which is of degree 4;*
(C4) *a $(3, 4^-, 4^-)$-vertex;*
(C5) *a light edge;*
(C6) *a light face.*

Proof. Assume to the contrary that the lemma is false and G is a counterexample. That is, G is a connected T-graph with $\Delta \geq 10$ and $\delta(G) \geq 2$ and containing none of the configurations (C1)–(C6). Since G contains no (C5), Claim 1 below holds automatically.

Claim 1. *Let $xy \in E(G)$ with $p(xy), q(xy) \geq 1$.*

(1) *If $d(x) = 2$, then $d(y) = \Delta$.*
(2) *Let $d(x) = 3$. If $q(xy) = 1$, then $d(y) \geq \Delta - 1$; If $q(xy) = 2$, then $d(y) = \Delta$.*