Doctopic: Algorithms, automata, complexity and games ARTICLE IN PRESS TCS:11215

Theoretical Computer Science ••• (••••) •••–•••

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Strong matching preclusion for non-bipartite torus networks ☆

Kai Feng

School of Computer and Information Technology, Shanxi University, Taiyuan, Shanxi 030006, PR China

ARTICLE INFO

Article history:
Received 11 January 2017
Received in revised form 2 June 2017
Accepted 2 June 2017
Available online xxxx
Communicated by S.-y. Hsieh

Keywords:
Interconnection networks
Perfect and almost perfect matchings
Strong matching preclusion
Torus

ABSTRACT

The strong matching preclusion number of a graph is the minimum number of vertices and edges whose deletion results in the remaining graph that has neither perfect matchings nor almost perfect matchings. The torus network is one of the most popular interconnection network topologies for massively parallel computing systems because of its desirable properties. It is known that bipartite torus networks have low strong matching preclusion numbers. Hu et al. [13] proved that non-bipartite torus networks with an odd number of vertices have good strong matching preclusion properties. To complete the study of strong matching preclusion problem for non-bipartite torus networks, in this paper, we establish the strong matching preclusion number and classify all optimal strong matching preclusion sets for the n-dimensional non-bipartite torus network with an even number of vertices, where $n \geq 3$.

1. Introduction

1.1. Problem description

A matching of a graph is a set of pairwise nonadjacent edges. For a graph with n vertices, a matching M is called perfect if its size $|M| = \frac{n}{2}$ for even n, or almost perfect if $|M| = \frac{n-1}{2}$ for odd n. A graph is matchable if it has either a perfect matching or an almost perfect matching. Otherwise, it is called not matchable. For graph-theoretical terminology and notation not defined here we follow [3]. A set F of edges in a graph G is called a matching preclusion set (MP set for short) if $G - F$ is not matchable. The matching preclusion number of G, denoted by $mp(G)$, is defined to be the minimum size of all possible such sets of G. The optimal MP set of G is any MP set whose size is $mp(G)$. This concept of matching preclusion was first presented by Brigham et al. [2] and further studied by [7,8,18]. It was introduced as a measure of robustness in the event of edge failure in interconnection networks, as well as a theoretical connection to conditional connectivity, "changing and unchanging of invariants" and extremal graph theory. We refer the readers to [2] for details.

The concept of strong matching preclusion was proposed by Park and Ihm [15]. A set F of vertices and/or edges in a graph G is called a strong matching preclusion set (SMP set for short) if $G - F$ is not matchable. The strong matching preclusion number of G, denoted by $smp(G)$, is the minimum cardinality of all SMP sets of G. The optimal SMP set of G is any SMP set whose size is $smp(G)$.

Specially, when G itself is not matchable, both $smp(G)$ and $mp(G)$ are regarded as zero. These numbers are undefined for a trivial graph with only one vertex. Notice that every MP set of a graph is a special SMP set of the graph. For any nontrivial graph G, we have $smp(G) \leq mp(G)$.

☆ This work is supported by the National Natural Science Foundation of China (No. 61502286).
E-mail address: fengkai@sxu.edu.cn.

http://dx.doi.org/10.1016/j.tcs.2017.06.002
0304-3975/© 2017 Elsevier B.V. All rights reserved.

When a set F of vertices and/or edges is removed from a graph, the set is called a fault set. For any vertex $v \in V(G)$, let $N_G(v)$ be all neighboring vertices adjacent to v and let $I_G(v)$ be all edges incident to v. Clearly, a fault set, which separates exactly one isolated vertex from the remaining graph with an even number of vertices, forms a simple SMP set of the original graph.

Proposition 1.1 *([15]). Let G be a graph. Given a fault vertex set $X(v) \subseteq N_G(v)$ and a fault edge set $Y(v) \subseteq I_G(v)$, $X(v) \cup Y(v)$ is an SMP set of G if (a) $w \in X(v)$ if and only if $(v, w) \notin Y(v)$ for every $w \in N_G(v)$, and (b) the number of vertices in $G - (X(v) \cup Y(v))$ is even.*

The above proposition suggests an easy way of building SMP sets. Any SMP set constructed as specified in Proposition 1.1 is called trivial. Let G be a graph. If $smp(G) = \delta(G)$, then G is called maximally strong matched. If every optimal SMP set of G is trivial, then G is called super strong matched. It is easy to see that, for an arbitrary vertex of degree at least one, there always exists a trivial SMP set which isolates the vertex. This observation leads to the following fact.

Proposition 1.2 *([15]). For any graph G with no isolated vertices, $smp(G) \leq \delta(G)$, where $\delta(G)$ is the minimum degree of G.*

1.2. Our motivation

Distributed processor architectures offer the advantage of improved connectivity and reliability. An important component of such a distributed system is the system topology, which defines the inter-processor communication architecture. This system topology forms the interconnection network. In certain application, every node requires a special partner at any given time and the matching preclusion number measures the robustness of this requirement in the event of link failures as indicated in [2]. Hence, in these interconnection networks, it is desirable to have the property that the only optimal MP sets and optimal SMP sets are those whose deletion gives an isolated vertex in the resulting graph. As an extensive form of matching preclusion, the problem of strong matching preclusion has been studied for many classes of interconnection networks [1,4–6,9–13,15–17].

Among various designs of large-scale networks, the Cartesian product method is a very effective method of building larger networks from several specified small-scale networks. Let $G = (V(G), E(G))$ and $H = (V(H), E(H))$ be two simple graphs. Their Cartesian product $G \square H$ is the graph with vertex set $V(G) \square V(H) = \{gh : g \in V(G), h \in V(H)\}$, in which two vertices g_1h_1 and g_2h_2 are adjacent if and only if $g_1 = g_2$ and $(h_1, h_2) \in E(H)$, or $(g_1, g_2) \in E(G)$ and $h_1 = h_2$. The n-dimensional torus network, denoted by $T(k_1, k_2, \ldots, k_n)$, constructed as the Cartesian product of cycles, is one of the most popular interconnection networks for massively distributed systems and has recently received significant attention [8, 12–14,16]. In [12] and [16], the strong matching preclusion problem was studied for two-dimensional torus networks and bipartite torus networks, and the following two results were given.

Theorem 1.1 *([12,16]). Let $k_1 \geq 3$ be an odd integer and let $k_2 \geq 3$ be an integer. Then $T(k_1, k_2)$ is maximally strong matched. Moreover, $T(k_1, k_2)$ is super strong matched if $k_2 \neq 4$.*

Theorem 1.2 *([16]). Let $k_1, k_2, \ldots, k_n$ be even integers with $k_i \geq 4$ for each $i = 1, 2, \ldots, n$. Then $smp(T(k_1, k_2, \ldots, k_n)) = 2$. Furthermore, each of its optimal SMP sets is a set of two vertices from the same partite set.*

Then, followed by the work above, Hu et al. [13] investigated the problem of strong matching preclusion for non-bipartite torus networks with an odd number of vertices and presented the following result.

Theorem 1.3 *([13]). Let $n \geq 3$ be an integer and $k_i \geq 3$ be an odd integer for every $1 \leq i \leq n$. Then $T(k_1, k_2, \ldots, k_n)$ is super strong matched.*

To complete the study of strong matching preclusion problem for n-dimensional torus networks, in this paper, we shall show that non-bipartite torus networks with an even number of vertices have good strong matching preclusion properties, and obtain the following result.

Theorem 1.4. *Let $n \geq 3$ be an integer and let $k_1, k_2, \ldots, k_n$ be integers with $k_i \geq 3$ for each $i = 1, 2, \ldots, n$. If not all the k_i's are even or odd, then $T(k_1, k_2, \ldots, k_n)$ is super strong matched.*

2. Preliminaries

Let C_k be the cycle of length k with the vertex set $V(C_k) = \{0, 1, \ldots, k-1\}$. Two vertices $u, v \in V(C_k)$ are adjacent in C_k if and only if $u = v \pm 1 \pmod{k}$. For clarity of presentation, we omit writing "mod k" in similar expressions for the remainder of the paper. The n-dimensional torus $T(k_1, k_2, \ldots, k_n)$ with $n \geq 2$ and $k_i \geq 3$ for all i is defined to be $C_{k_1} \square C_{k_2} \square \cdots \square C_{k_n}$