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Higher-level constructs for families and multisets

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ABSTRACT

In an earlier paper we argued that the concept of multisets should be based on families, that is, on sets and functions between sets, rather than, as usual, on sets and cardinal numbers. We showed how certain fundamental problems regarding the distinguishablity of objects as well as unexpected anomalies of the basic operations of union, intersection, and complement can be avoided elegantly using families rather than multisets. On the other hand, there is a trivial mapping of families to multisets. Hence, using families does not introduce any significant formal complications.

The difficulties with the usual definition of multisets by multiplicities reach beyond the philosophical or metamathematical foundations. For the basic operations one can find acceptable compromises. For higher-level set constructs like Cartesian products, projections, relations, functions, or the power set the multiset counterparts are rather contrived, and many of the constructions leave the strict definition of multisets, actually introducing family-like constructs through a back door.

In the earlier paper we proposed to use families instead of multisets to resolve the basic problems. In this paper we show that families instead of multisets support the higher-level constructions even, when no appropriate multiset-based constructions exist. We also show that multisets form a category and that the natural mapping from families to multisets is a functor. This emphasizes our claim that, for a definition of multisets, one should start with families and only introduce multiplicities as a secondary concept.

1. Background

For reasons not relevant to the present paper we scrutinized the commonly accepted definition of multisets in [9,10]. We arrived at the conclusion that, as a basic mathematical concept, multisets are inadequate and should be replaced by that of families. In that work, the case was argued on metamathematical grounds. We showed that there is a functor-like mapping which translates results about families into results about multisets. We only provided as much detail as was needed for the argument that families are to be preferred over multisets. In particular, we showed that fundamental problems with operations like union, intersection, and complement in multiset theory can be solved elegantly within the theory of families.

There are higher-level constructs on multisets like “powersets”, Cartesian products, projections etc. which we omitted in [9], but which would deserve a careful examination, too, if only for the reason that the relevant family-theoretic constructs seem not have been defined. For multisets most higher-level constructions solely based on multiplicities have failed;

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escape routes have been taken as in [2], which just implies that the standard definition of multisets does not support a formulation of a self-contained mathematical theory. On the other hand, for families these constructions are straightforward, and the connection to multisets proper is established by a simple mapping. Hence our basic procedure is as follows:

1. We review multiset-constructions according to the literature.
2. We define their intuitive analogues in the theory of families.
3. We examine the extent to which the mapping from families to multisets captures the intended meaning.

There does not seem to be a theory of families even though the concept is used extensively – we invent part of the theory as needed.

In the earlier paper we used the term of "convention" to denote definitions or postulates which might be contentious due their metamathematical nature. In the present paper most of these are called "definitions". The difference is philosophical and not relevant in the present context.

2. Notation and some basic notions

By $\mathbb{N}$ we denote the set of positive integers; $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. For $m, n \in \mathbb{N}_0$, the difference $m \dot{-} n$ is defined as $m - n$ for $m \geq n$ and as 0 for $m < n$.

We use standard set theoretic notation. In particular, the power set of a set S is denoted by $\mathsf{P}S$. We shall apply the operator P also to other set-like constructs as families or multisets (to be defined further below). To emphasize the fact that the result is a set we shall use $\mathsf{P}^{\mathsf{set}}$ instead of P when needed. The cardinality of a set S is denoted by $|S|$. When S is a subset of a set X, the complement of S with respect to X is the set $X \setminus S$. The disjoint union of two sets S and T is denoted by $S \,\dot{\cup} T$. For any objects x and y, $\langle x, y\rangle$ denotes the ordered pair composed of x and y. Consider two sets S and T. Then $S \times T = \{\langle s, t\rangle \mid s \in S, t \in S\}$ is the direct (or Cartesian) product of S and T. A relation f from S to T is a subset of $S \times T$. For a relation f and an element $s \in S$, $f(s) = \{t \mid t \in T, \langle s, t\rangle \in f\}$. The inverse of a relation f is the set $f^{-1} = \{\langle t, s\rangle \mid t \in T, s \in S, \langle s, t\rangle \in f\}$. Thus, for $t \in T$, $f^{-1}(t) = \{s \mid s \in S, \langle s, t\rangle \in f\}$. A relation f is a mapping, if $f(s)$ is a singleton set for all $s \in S$. It is an injective mapping, if also f^{-1} is a mapping. When f is a mapping, the graph of f, $\mathsf{graph}\, f$, is the relation defining f.

Consider a mapping $S \to T$. For a set U and a mapping $g : T \to U$, the composite mapping[1] $f \circ g$ is the mapping $h : S \to U$ defined by $h(s) = g(f(s))$ for all $s \in S$.

In the sequel, whenever it is convenient and not compromising accuracy, we do not distinguish notationally between singleton sets and their elements. Thus, if f as above is not just a relation, but even a mapping, $f(s) = \{t\}$ would be written as $f(s) = t$.

In addition to the usual assumptions for set theory we postulate that there is a universe.

Definition 1 *(Grothendieck, [6]).* A *universe* is a non-empty set U with the following properties:

1. if $x \in U$ and $y \in x$ then $y \in U$;
2. if $x, y \in U$ then also $\{x, y\} \in U$;
3. if $x \in U$ then also $\mathsf{P}x \in U$;
4. if $\{x_\alpha\}_{\alpha \in I}$ is a family such that $I \in U$ and $x_\alpha \in U$ for all $\alpha \in I$, then $\bigcup_{\alpha \in I} x_\alpha \in U$.

The elements of U are called U-sets; the subsets of U are called U-classes. Throughout this article we assume a fixed universe U with $\mathbb{N}_0 \in U$, with all sets under consideration being elements of this universe, and with all classes considered being subsets of u; hence, by 'set' we mean 'U-set', by 'class' we mean 'U-class'.

Without special mention, we use the axiom of choice whenever required. Further notation and notions will be introduced as needed.

3. Multisets and their properties

We provide the definition of multisets and list their basic properties. This material appears in some form or other in nearly every publication concerning multisets or their applications. Hence we do not provide specific references.

Definition 2 *(Multiset).* Let X be a set. A *multiset* over X is a mapping $\mu : X \to \mathbb{N}_0$.

One finds many equivalent variants of this definition in both the literature on multisets and the literature using multisets. A quite common one reads as follows: A multiset over X is a set of pairs $(x, \mu(x))$ with $x \in X$ and $\mu(x) \in \mathbb{N}_0$. Obviously this

[1] The order of the factors in a composition varies in the literature. We use the order which is consistent with the composition of relations.