Theoretical Computer Science 685 (2017) 83–103

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

On a graph calculus for modalities ☆

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ARTICLE INFO

Article history:
Received 12 July 2015
Received in revised form 23 October 2016
Accepted 22 November 2016

Keywords:
Modal logics
Graph calculus
Kripke semantics
Special modalities
Frames

ABSTRACT

We present a sound and complete graph calculus for modalities. This calculus is a general framework for expressing modal formulas and frame properties, with a rich repertoire of relations, and reasoning about them in a uniform manner. The calculus employs graphical interpretations of logical operators and builds graphical objects that represent conditions on Kripke structures.

1. Introduction

We present a sound and complete graph calculus for (classical) multi-modal logics [3,6,7].

We begin with some ideas about graph calculi with the purpose of clarifying their role in modal logics. The formalism of graph calculus was originally conceived to handle graphically binary relations and their operations [8,12]. It can also be used as a graphical (meta-)language for modal logics, providing an intuitive and natural way of handling formulas and frame properties.

For instance, the fact that formula $\langle r \rangle\, p$ holds at u is represented by $\hat{u} \xrightarrow{r} w - - \prec p$.[1] This amounts exactly to the model one would have in mind upon first seeing this modal formula, i.e. p holds at some w r-reachable from u. Similarly, the fact that $\langle r \rangle \langle r \rangle\, p$ holds at u can be represented by $\hat{u} \xrightarrow{r} v \xrightarrow{r} w - - \prec p$. Also, we represent that $\langle r \rangle \langle s \rangle (p \wedge q)$ holds at u by $\hat{u} \xrightarrow{r} v \xrightarrow{s} w \begin{smallmatrix} - - \prec p \\ - - \prec q \end{smallmatrix}$. In this manner, we display the conditions for satisfaction of a modal formula: we will write $\langle t \rangle\, \varphi$ holds at x as $\hat{x} \xrightarrow{t} y - - \prec \varphi$.

The aim of this paper is to provide a framework for representing and reasoning graphically about modal logics [3,6,7]. Now, what are the advantages of presenting a modal logic as a graph calculus? Our graph languages are more expressive than standard ways of representing modal operators. Modal languages correspond to the guarded fragment of First-order

☆ This work was supported by the Brazilian research agencies CNPq grants 303740/2015-9, 308565/2014-2 and 444645/2014-4 and CAPES. Helpful comments by the referees are gratefully acknowledged.

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[1] We use $\hat{}$ to mark the present node; see also Section 2 (Graphs and Modalities: Basic Ideas), p. 84.

http://dx.doi.org/10.1016/j.tcs.2016.11.037
0304-3975/

Logic, whereas graph languages correspond to the fragment of First-order Logic with at most two free variables, unary and binary predicates and with universal and existential quantifiers, a very expressive fragment. We will not need the whole fragment to present modal operators, but the graph calculus will allow us to express some operators that cannot be expressed in standard ways of presenting modal logics: modalities like intersection [1,2] and difference [3] can be, easily and naturally, presented without any side conditions, in contrast with other presentations [1–3].

We would like to clarify some distinctions between graph calculi and other methods for handling logics. Natural deduction relies on rules for introducing and eliminating logical operators (connectives, etc.) and its aim is building derivation trees. In tableaux, the emphasis is on rules that describe truth/falsity conditions for logical operators and the aim is constructing refutation trees. Graph calculi employ graphical interpretations of logical operators and the aim is building graphical objects that represent conditions on models.

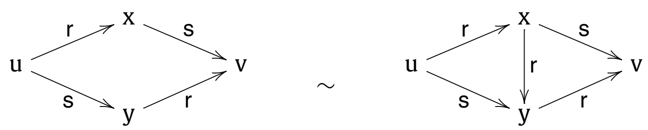
To illustrate the ideas discussed above, consider the modal operator $\langle \mathsf{r} \sqcap \mathsf{s} \rangle$, which involves the intersection of the relations R and S corresponding to r and s, respectively. Its intended meaning is: $\langle \mathsf{r} \sqcap \mathsf{s} \rangle \varphi$ holds at a state a iff φ holds at some state b that is $\mathsf{R} \cap \mathsf{S}$-reachable from a (i.e. $\mathfrak{M}, a \Vdash \langle \mathsf{r} \sqcap \mathsf{s} \rangle \varphi$ iff there exists b such that $(a, b) \in \mathsf{r}^{\mathfrak{M}} \cap \mathsf{s}^{\mathfrak{M}}$ and $\mathfrak{M}, b \Vdash \varphi$). One could try to present it axiomatically by $\langle \mathsf{r} \sqcap \mathsf{s} \rangle \varphi \leftrightarrow \langle \mathsf{r} \rangle \varphi \wedge \langle \mathsf{s} \rangle \varphi$. Of course, this does not work. The right-hand side amounts to $\langle \mathsf{r} \rangle \varphi \wedge \langle \mathsf{s} \rangle \varphi$ holds at state a iff there exist states b' and b'' such that b' is R-reachable from a with φ holding at b' and b'' is S-reachable from a with φ holding at b''; the difficulty lies in forcing $b' = b''$. (Formula $\langle \mathsf{r} \sqcap \mathsf{s} \rangle \varphi \rightarrow \langle \mathsf{r} \rangle \varphi \wedge \langle \mathsf{s} \rangle \varphi$ is valid, but one can easily falsify $\langle \mathsf{r} \rangle \varphi \wedge \langle \mathsf{s} \rangle \varphi \rightarrow \langle \mathsf{r} \sqcap \mathsf{s} \rangle \varphi$ at state a of the model with $M = \{a, b', b''\}$, $\mathsf{R} = \{(a, b')\}$, $\mathsf{S} = \{(a, b'')\}$ and $\mathbb{V}(\mathrm{p}) = \{b', b''\}$.) One can similarly see that $[\mathsf{r} \sqcap \mathsf{s}] \varphi \leftrightarrow [\mathsf{r}] \varphi \wedge [\mathsf{s}] \varphi$ does not work. For tableaux, we could have rules like the following ones:

$$\mathrm{R}_{\langle \mathsf{r} \sqcap \mathsf{s} \rangle}: \quad \frac{\mathsf{u} : \langle \mathsf{r} \sqcap \mathsf{s} \rangle \varphi}{\mathsf{v} : \varphi} \qquad\qquad \mathrm{R}_{\neg \langle \mathsf{r} \sqcap \mathsf{s} \rangle}: \quad \frac{\mathsf{u} : \neg \langle \mathsf{r} \sqcap \mathsf{s} \rangle \varphi}{\mathsf{v} : \neg \varphi}$$

But, we need some sort of side conditions connecting u and v (see [1,2]).

In contrast, in graph calculus we can have a natural rule defining $\mathsf{r} \sqcap \mathsf{s}$, namely: rewrite $\mathsf{u} \xrightarrow{\mathsf{r} \sqcap \mathsf{s}} \mathsf{v}$ as $\mathsf{u} \underset{\mathsf{s}}{\overset{\mathsf{r}}{\rightrightarrows}} \mathsf{v}$. This parallel-arc rule, together with the above rule for $\langle \mathsf{t} \rangle \varphi$, will enable us to write the fact that formula $\langle \mathsf{r} \sqcap \mathsf{s} \rangle \varphi$ holds at x as $\widehat{\mathsf{x}} \underset{\mathsf{s}}{\overset{\mathsf{r}}{\rightrightarrows}} \mathsf{y} - - \!\!\prec\! \varphi$.

Now, consider a more general situation. Assume that one wishes to express the condition: "whenever we have $c\,\mathsf{R}\,a$, $a\,\mathsf{S}\,d$, $c\,\mathsf{S}\,b$, $b\,\mathsf{R}\,d$, we also have $a\,\mathsf{R}\,b$". This can be expressed quite neatly in graph calculus by the following natural rewrite rule:



It is not easy to figure out how one could express this situation by modal formulas, in tableaux, natural deduction or axiomatically. The 2-dimensional notation used in the graph approach has various advantages: besides its rich expressive power, it has a visual appeal that renders expressing formulas properties and reasoning about them both natural and easy.

In the sequel we will present our (rather expressive and easy to use) graph calculus for modalities [3,6,7]. In Section 2 we introduce, by means of illustrative examples, some ideas about graphs and modalities. In Section 3 we examine some concepts and results about graphs. Section 4 presents our calculus, which we analyze in Section 5. Section 6 presents some concluding remarks and comparison with other approaches, whereas Appendix A gives some details. We try to work at two levels: an intuitive one and a formal one. The intuitive level illustrates how the calculus is used by relying on visual examples. At the other level, which may be understood as a meta-level, we describe the calculus in a general manner, by means of precise concepts.

2. Graphs and modalities: basic ideas

We now introduce informally some basic ideas about graphs and modalities. These and other ideas will be formulated more precisely later on: in Sections 3 and 4. We use terminology inherited from previous papers, which should not be confused with similar terms in other areas.

A graph amounts to a finite set of (alternative) slices. A slice S consists of an underlying draft $\underline{\mathrm{S}}$ together with a list of distinguished nodes. A draft consists of finite sets of nodes and arcs. In the usual graph-theoretic terms, a draft is a labeled graph and a slice is a one or two rooted graph. Drafts (and sketches, to be introduced in Section 3) serve to describe restrictions on states. Slices may be 1-ary or 2-ary: a 1-ary slice has a single distinguished node (marked $\widehat{\ }$) and a 2-ary slice has a pair of distinguished nodes (marked by $\rightarrow$); they will represent sets of states and sets of pairs of states, respectively (see Examples 2.1: Formula consequence and 2.4: Relational inclusions).