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On mixed polynomials of bidegree $(n, 1)$

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ABSTRACT

Specifying the bidegrees (n, m) of mixed polynomials $P(z, \bar{z})$ of the single complex variable z, with complex coefficients, allows to investigate interesting roots structures and counting; intermediate between complex and real algebra. Multivariate mixed polynomials appeared in recent papers dealing with Milnor fibrations, but in this paper we focus on the univariate case and $m = 1$, which is closely related to the important subject of harmonic maps. Here we adapt, to this setting, two algorithms of computer algebra: Vandermonde interpolation and a bissection-exclusion method for root isolation. Implemented in Maple, they are used to explore some interesting classes of examples.

1. Introduction

An expression $P(z, \bar{z}) = \sum_{k=0..n} \sum_{j=0..m} a_{k,j} z^k \bar{z}^j$ where z and $\bar{z}$ are complex conjugated, is called a (univariate) mixed polynomial of bidegree (n, m). We will assume $m \leq n$ and concentrate on the case where $m = 1$. Our aim is to study the roots in $\mathbb{C}$ of P. Identifying $\mathbb{C}$ with $\mathbb{R}^2$ and separating real and imaginary parts of P, i.e. writing $P = f(x, y) + ig(x, y)$ with $i^2 = -1$ and $z = x + iy$, we get a pair of real bivariate polynomials of degrees at most $n + m$. Conversely from a pair of bivariate polynomials $(f(x, y), g(x, y))$, letting $x = \frac{z+\bar{z}}{2}$, $y = \frac{z-\bar{z}}{2i}$ and $P = f + ig$, we get a univariate mixed polynomial. However, since the two representations are different, we can investigate interesting roots structures and develop algorithms, intermediate between complex and real algebra. This representation can be also used with several variables $(z_1, .., z_l)$; it received a renewed interest with the works in Algebraic Geometry of [17] on a new exotic sphere (à la Pham–Brieskorn); more recently Mutsuo Oka [12], thanks to mixed polynomials, answered a question of [10] on real generalizations of Milnor fibration theorem. Harmonic polynomials and rational maps are important special cases of mixed polynomials; they have been extensively studied and were applied to the study of gravitational lensing [7,14]. Indeed after simplification, the roots finding problem for a mixed polynomial equation $P(z, \bar{z})$ of bidegree$(n, 1)$ reduces to the study of $\bar{z} = r(z)$, where r is a rational map; we will briefly recall some recent root counting formulas obtained in that field, [6,20,7,16,2].

Several techniques developed in Computer algebra seem useful to better investigate these objects. Specially in the case $m = 1$ where one expects properties similar to those of usual complex univariate polynomials. Unfortunately, the presentation of a univariate polynomial as a product via its roots is not valid in this context. Moreover, although P of bidegree $(n, 1)$ has $2n + 2$ coefficients, it may admit more than $2n + 2$ roots in $\mathbb{C}$. We will discuss and illustrate this behavior, directly related to bounding the number of zeros of harmonic maps. Beside the case $m = 1$, the results obtained so far on harmonic polynomials, see e.g. [24,21,8], concentrated on m near n. The study of the case $m = 2$ is still lagging behind.

In this paper, we adapt two basic algorithms in this new setting and use them to explore some interesting classes of examples, including random mixed polynomial of bi-degree $(n, 1)$ for rather large n. The first tool is a variant of Vandermonde

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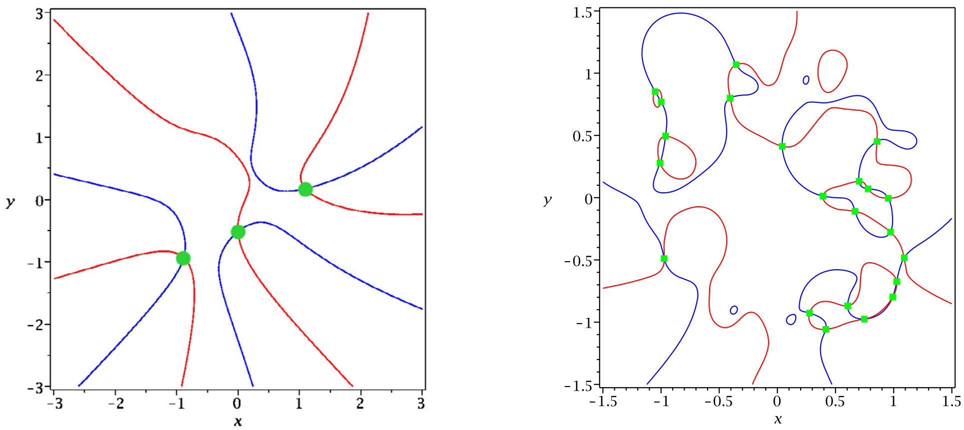

Fig. 1. Examples 1 and 2. (For interpretation of the references to color in this figure, the reader is referred to the web version of this article.)

matrix needed to interpolate $P(z, \bar{z})$, in such a way that we can prescribe some roots in $\mathbb{C}$ and then investigate the set of roots of P in $\mathbb{C}$. The second tool is a bisection-exclusion method which generalizes the classical one, see e.g. [23]. Together with a specific Newton process, it allows us to certify the set of complex roots we computed in each of our examples. We do not provide general complexity formulas but restrict ourselves to the case of mixed polynomials with simple roots (with an algorithm to check this property). Experiments, with the computer algebra system Maple, on mixed polynomials, of degrees $(n, 1)$, with given random distribution of coefficients allowed to observe interesting patterns.

The paper is organized as follows: In the next section 2, after some examples we present general properties of mixed polynomials and give an overview of results recently obtained on zeros counting of rational harmonic maps. In section 3, we construct generalized Vandermonde matrices and prove that they are generically invertible. In section 4, we present some investigation tools and together with examples; we investigate the effect of choosing the coefficients with several stochastic distributions. In section 5, we develop for the case $(n, 1)$, our bisection-exclusion method for locating the roots of P in $\mathbb{C}$, together with a Newton process and a test to check that a small disc contains only one root.

This paper is an amplification and a continuation of our presentation at the conference SNC'2014 [4].

We denote by $\bar{a}$ the complex conjugated of a complex number a, and by $\bar{P}$ the complex conjugated of a (mixed or usual) polynomial P, its coefficients are the complex conjugated of the coefficients of P.

2. General properties

We begin with some examples of mixed polynomials and pictures of their roots.

Example 1. A random mixed polynomial of bidegree $(4, 1)$

$$P := (4 - 3i)z^4\bar{z} + (3 + 7i)z^4 + (8i)z^3\bar{z} + (7 + 9i)z^3 + (-6 - 9i)z^2\bar{z}$$
$$+ (6 - 3i)z^2 + (-5 - 6i)z\bar{z} + (1 - 7i)z + (-5 - 9i)\bar{z} + 4 + 2i.$$

It has 3 roots in $\mathbb{C}$ shown in green in Fig. 1 left. Writing $P = f(x, y) + ig(x, y)$, the implicit curves defined by $f = 0$ and $g = 0$ are shown in red and blue.

Example 2. An example of a random polynomial of bidegree $(17, 15)$ with 19 roots, see Fig. 1 right.

Example 3. Consider $P = z\bar{z} + e$, when $e = -1$, its roots form a circle; when $e = 0$, the only root is a point; while when $e = 1$, P has no root in $\mathbb{C}$.

We briefly review some properties of univariate mixed polynomials inherited by their representations.

2.1. Factorization

The product of two mixed polynomials $P_3 = P_1 P_2$ can be expressed by a set of algebraic conditions on their coefficients, identical to the set of conditions corresponding to “usual” bivariate polynomials with the same bidegrees. Therefore, the factorization properties and algorithms valid for bivariate polynomials, are also valid for univariate mixed polynomials.

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