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Witness to non-termination of linear programs

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ABSTRACT

In his CAV 2004 paper, Tiwari has proved that, for a class of linear programs over the reals, termination is decidable. And Tiwari has shown that the termination of a linear program $\mathbf{P}_1$ whose assignment matrix $\tilde{A}$ is not in the Jordan canonical form is equivalent to that of a linear program $\mathfrak{J}_1$, whose assignment matrix A is in the Jordan Canonical Form. In most cases, the method of Tiwari provides only a so-called N-nonterminating point. In this paper, we propose two new methods to decide whether Program $\mathbf{P}_1$ terminates or not over the reals. Our methods are based on the construction of a subset of the set NT of non-terminating points of Program $\mathfrak{J}_1$. Any point in such a subset is a witness to non-termination of Program $\mathfrak{J}_1$. Furthermore, it is shown that Program $\mathfrak{J}_1$ is non-terminating if and only if such a subset is nonempty. In terms of the property, the first method is given to check whether Program $\mathfrak{J}_1$ terminates or not. Different from the existing methods, the point obtained by our first method is a non-terminating point, rather than a N-nonterminating point. More importantly, such a subset is also proven to be $A^{\hat{B}}$-invariant for some positive integer $\hat{B}$. This enables us to check directly the termination of Program $\mathfrak{J}_1$ by verifying the satisfiability of finitely many quantified formulas over the reals. This suggests our second method for checking the termination of Program $\mathfrak{J}_1$.

1. Introduction

It is well known that guaranteeing software systems trustworthy is a grand challenge in theoretical computer science [1–3]. As one of the building blocks of automated program verification, termination analysis has attracted increasing interest in the recent years. However, the termination problem is undecidable in most cases. Therefore, most well-established work concentrates on the construction of well-founded ranking functions [4–9]. Especially, Poldelski and Rybalchenko [9] first presented a complete method for the synthesis of linear ranking functions in 2004. But, it has been shown that the existence of ranking functions is just a sufficient (but not necessary) condition for guaranteeing the termination of loops. That is to say, one can construct an example of a loop that terminates but has no ranking function. Because of the reasons mentioned above, people pay attention to explore a decidable class of loops. For example, Tiwari [10] in 2004 showed that the termination of a linear loop program of the following shape is decidable over the reals, as follows:

$$\mathbf{P_1} \quad \text{while } (\tilde{B}X > 0)\ \{X := \tilde{A}X\}$$

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where $\tilde{B} \in \mathbb{R}^{s\times L}$ is called the condition matrix, and $\tilde{A} \in \mathbb{R}^{L\times L}$ is called the assignment matrix. This classic work shows us new insight on termination problem for programs. It has been pointed out that although a linear program may not be presented in this form, termination problem can always be reduced to this form by [10]. In Tiwari's method, the termination of $\mathbf{P}_1$ is reduced equivalently to that of $\mathfrak{J}_1$ whose assignment matrix is in the Jordan canonical form. In 2006, Braverman [11] generalized the work of Tiwari and proved that the loops of the above class is also decidable over the integers. But, the methods of Tiwari and Braverman do not consider how to get a witness to nontermination. Following their work, Xia et al. [12] considered the termination of a more general class of loops with nonlinear constraints and linear updates. They proved that under proper conditions, such loops were decidable over the reals. Since the decision procedure given by Tiwari depends on the computation of Jordan canonical forms, Yang et al. [13] presented a purely symbolic method to compute Jordan canonical forms. In addition, under the assumption that $\tilde{A}$ is diagonalizable matrix and its all eigenvalues are real, Rebiha et al. studied the termination of $\mathbf{P}_1$ and presented a method of generating the set of N-nonterminating points of $\mathbf{P}_1$ in [14,15]. Recently, Ouaknine et al. [16] show decidability of termination of simple linear loops over the integers under the assumption that the assignment matrix is diagonalizable. And the work of Ouaknine et al. is the first substantial advance on an open problem of Braverman [11].

In [17], we reconsider the same termination problem proposed and analyzed by Tiwari in 2004, and present a recursive algorithm for the termination of program $\mathbf{P}_1$. For clarity, we describe below the main ideas presented in [17] briefly. First, we reduce $\mathbf{P}_1$ to $\mathfrak{J}_1$ by the computation of the Jordan canonical form of the assignment matrix $\tilde{A}$ of $\mathbf{P}_1$. Since the assignment matrix A of $\mathfrak{J}_1$ is in the Jordan canonical form, we present two methods to check the termination of two special classes of linear programs, according to the number of Jordan blocks in A. Namely, we give a simple method to decide the termination of a special class of linear loops whose assignment matrices consist only of one Jordan block with positive real eigenvalue, i.e., $A = J(\lambda), \lambda > 0$. Furthermore, for these special loops, we construct a subset of the set of nonterminating points, which enables us to analyze the termination of this kind of loops only by determining whether the subset is empty or not. This result can also be generalized to determine the termination of another special class of linear programs, whose assignment matrices consist only of finitely many Jordan blocks with the same eigenvalue. Second, for the general program $\mathfrak{J}_1$ whose assignment matrix is $A = diag(J_1(\lambda_1), ..., J_s(\lambda_s))$, $\lambda_i > 0$, a recursive decision process, which reduces the termination of the general class of programs to that of the above mentioned two special classes of programs, is developed to analyze the termination of $\mathfrak{J}_1$ in [17].

In this paper, we will show that for the general Program $\mathfrak{J}_1$ whose assignment matrix is in the Jordan canonical form, i.e., $A = diag(J_1(\lambda_1), ..., J_s(\lambda_s))$, $\lambda_i > 0$, a subset of the set NT of its nonterminating points can still be constructed. It will be shown that such a constructed subset has two properties:

- Program $\mathfrak{J}_1$ is nonterminating over the reals if and only if such a subset is not empty.
- The constructed subset is $A^{\hat{B}}$-invariant for some positive integer $\hat{B}$.

The above two properties suggest two methods for checking the termination of Program $\mathfrak{J}_1$, respectively. Clearly, they also suggest two methods for checking the termination of $\mathbf{P}_1$, since the termination of $\mathbf{P}_1$ is equivalent to that of $\mathfrak{J}_1$. By the first property, Checking if Program $\mathbf{P_1}$ terminates is equivalent to checking if the constructed subset is empty. And such a subset can be characterized by semi-algebraic systems. Therefore, for Program $\mathbf{P}_1$, in our first method, we first need to reduce $\mathbf{P}_1$ to $\mathfrak{J}_1$ by computing the Jordan canonical form of the assignment matrix $\tilde{A}$ of $\mathbf{P}_1$. And then, we construct the desired subset of the set NT of non-terminating points of $\mathfrak{J}_1$ and check whether such a subset is empty or not. Different from the methods given by Tiwari, Braverman and Rebiha et al., any point in such a subset must be a non-terminating point, rather than a N-nonterminating point. In addition, our first method is different from the method given in [17], since the latter is a recursive procedure. Besides, by the second property as above, if the constructed subset is not empty, then there exists an $A^{\hat{B}}$-invariant set, which can be expressed as a quantified formula over the theory of linear arithmetic interpreted over the reals, in the region specified by the loop conditions of Program $\mathfrak{J}_1$. This suggests the second decision method for the termination of $\mathbf{P}_1$. Also, our second method is different from Theorem 3 in [10], since Theorem 3 given by Tiwari just can deal with the termination of two variables loops. The reason is that Tiwari's Theorem 3 depends on the fact that in 2-dimensional case, the set NT of nonterminating points of $\mathbf{P_1}$ will be an $\tilde{A}$-invariant sector and it can be specified by its two boundary rays. It has been pointed out by Tiwari that Theorem 3 in [10] can not be generalized to higher dimensions since the region NT in high-dimensional case may not be specified by finitely many hyperplane boundaries. In contrast, our methods do not need to construct NT, but just needs to construct a subset of NT, which can be specified by semi-algebraic systems consisting of finitely many inequalities and equalities.

The rest of the paper is organized as follows. In Section 2, we recall some important results presented in [10]. In Section 3, for Program $\mathfrak{J}_1$, we construct a subset of the set NT of non-terminating points of Program $\mathfrak{J}_1$ and prove that such a subset has the two properties as above. In Section 4, an example is given to illustrate our methods. Finally, we conclude the paper in Section 5.

2. Previous results

In [10], Tiwari establishes the decidability of the termination problem for linear loops of the form $\mathbf{P_1}$. Generally speaking, we say that Program $\mathbf{P_1}$ is nonterminating over the reals, if there is a point $X \in \mathbb{R}^L$, such that $B\tilde{A}^n X > 0$ holds for all $n \geq 0$.