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Nearly optimal computations with structured matrices

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ABSTRACT

We estimate the Boolean complexity of multiplication of structured matrices by a vector and the solution of nonsingular linear systems of equations with these matrices. We study four basic and most popular classes, that is, Toeplitz, Hankel, Cauchy and Vandermonde matrices, for which the cited computational problems are equivalent to the task of polynomial multiplication and division and polynomial and rational multipoint evaluation and interpolation. The Boolean cost estimates for the latter problems have been obtained by Kirrinnis in [10], except for rational interpolation. We supply them now as well as the Boolean complexity estimates for the important problems of multiplication of transposed Vandermonde matrix and its inverse by a vector. All known Boolean cost estimates from [10] for such problems rely on using Kronecker product. This implies the d-fold precision increase for the d-th degree output, but we avoid such an increase by relying on distinct techniques based on employing FFT. Furthermore we simplify the analysis and make it more transparent by combining the representations of our tasks and algorithms both via structured matrices and via polynomials and rational functions. This also enables further extensions of our estimates to cover Trummer's important problem and computations with the popular classes of structured matrices that generalize the four cited basic matrix classes, as well as the transposed Vandermonde matrices. It is known that the solution of Toeplitz, Hankel, Cauchy, Vandermonde, and transposed Vandermonde linear systems of equations is generally prone to numerical stability problems, and numerical problems arise even for multiplication of Cauchy, Vandermonde, and transposed Vandermonde matrices by a vector. Thus our FFT-based results on the Boolean complexity of these important computations could be quite interesting because our estimates are reasonable even for more general classes of structured matrices, showing rather moderate growth of the complexity as the input size increases.

1. Introduction

Table 1 displays four classes of most popular structured matrices, which are omnipresent in modern computations for Sciences, Engineering, and Signal and Image Processing. These basic classes have been naturally extended to the four larger classes of matrices, $\mathcal{T}$, $\mathcal{H}$, $\mathcal{V}$, and $\mathcal{C}$, that have structures of Toeplitz, Hankel, Vandermonde and Cauchy types, respectively. They include many other important classes of structured matrices such as the products and inverses of the matrices of these

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Table 1
Four classes of structured matrices.

Toeplitz matrices $T = (t_{i-j})_{i,j=0}^{n-1}$ $\begin{pmatrix} t_0 & t_{-1} & \cdots & t_{1-n} \\ t_1 & t_0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & t_{-1} \\ t_{n-1} & \cdots & t_1 & t_0 \end{pmatrix}$	Hankel matrices $H = (h_{i+j})_{i,j=0}^{n-1}$ $\begin{pmatrix} h_0 & h_1 & \cdots & h_{n-1} \\ h_1 & h_2 & ⋰ & h_n \\ \vdots & ⋰ & ⋰ & \vdots \\ h_{n-1} & h_n & \cdots & h_{2n-2} \end{pmatrix}$
Vandermonde matrices $V = V_{\mathbf{s}} = \left(s_i^{j-1}\right)_{i,j=1}^{n}$ $\begin{pmatrix} 1 & s_1 & \cdots & s_1^{n-1} \\ 1 & s_2 & \cdots & s_2^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & s_n & \cdots & s_n^{n-1} \end{pmatrix}$	Cauchy matrices $C = C_{\mathbf{s},\mathbf{t}} = \left(\frac{1}{s_i - t_j}\right)_{i,j=1}^{n}$ $\begin{pmatrix} \frac{1}{s_1-t_1} & \cdots & \frac{1}{s_1-t_n} \\ \frac{1}{s_2-t_1} & \cdots & \frac{1}{s_2-t_n} \\ \vdots & & \vdots \\ \frac{1}{s_n-t_1} & \cdots & \frac{1}{s_n-t_n} \end{pmatrix}$

four basic classes, as well as the companion, Sylvester, subresultant, Loewner, and Pick matrices. All these matrices can be readily expressed via their displacements of small ranks [15, Chapter 4], which implies their further attractive properties:

- Compressed representation of matrices as well as their products and inverses through a small number of parameters.
- Multiplication by a vector in nearly linear arithmetic time.
- Solution of nonsingular linear systems of equations with these matrices in quadratic or nearly linear arithmetic time.

These properties enable efficient computations, closely linked and frequently equivalent to fundamental computations with polynomials and rational functions, in particular to the multiplication, division, multipoint evaluation and interpolation [19]. Low arithmetic cost is surely attractive, but substantial growth of the computational precision quite frequently affects the known algorithms having low arithmetic cost (see, e.g., [5]). So the estimation of the complexity under the Boolean model is more informative, although technically more demanding.

To the best of our knowledge, the first Boolean complexity bounds for multipoint evaluation are due to Ritzmann [21]. We also wish to cite the papers [28] and [12], although their results have been superseded in the advanced work of 1998 by Kirrinnis [10], apparently still not sufficiently well known. Namely in the process of studying approximate partial fraction decomposition he has estimated the Boolean complexity of the multipoint evaluation, interpolation, and the summation of rational functions. He required the input polynomials to be normalized, but actually this was not restrictive at all. For simplicity we assume the evaluation at the points of small magnitude, but our estimates can be rather easily extended to the case of general input. Kirrinnis' study as well as all previous estimates of the Boolean complexity of these computational problems rely on multiplying polynomials as integers, by using Kronecker's product, aka binary segmentation, as proposed in [7]. This implies the d-fold increase of the computational precision for the d-th degree output. In contrast our results rely on FFT-based algorithms for multiplying univariate polynomials and avoid this precision growth. This does not lead to an improvement of the complexity bounds, but allows us to perform polynomial operations without relying solely on algorithms for fast multiplication of long integers, which are only efficient when the precision of computing grows large.

We represent our FFT-based estimates and algorithms in terms of operations both with structured matrices and with polynomial and rational functions. In both representations the computational tasks and the solution algorithms are equivalent, and so the results of [10] for partial fraction decomposition can be extended to most, although not all, of these tasks. By using both representations, however, we make our analysis more transparent. Furthermore in Section 7 we extend Kirrinnis' results to the solution of a Cauchy linear system of equations (which unlike [10] covers rational interpolation) and in Section 7.2 to the solution of Trummer's celebrated problem [8,9,6], having important applications to mechanics (e.g., to particle simulation) and representing the secular equation, which is the basis for the MPSolve, the most efficient package of subroutines for numerical polynomial root-finding [3].

Our estimates cover multiplication of the matrices of the four basic classes of Table 1 by a vector and solving Vandermonde and Cauchy linear systems of equations. (As we mentioned, these tasks are closely linked and frequently equivalent to the listed tasks of the multiplication, division, multipoint evaluation and interpolation of polynomials and rational functions.) Expressing the solution of these problems in terms of matrices has a major advantage: it can be extended to matrices of the four larger matrix classes $\mathcal{T}$, $\mathcal{H}$, $\mathcal{V}$, and $\mathcal{C}$. We specify these extensions in the last sections of the paper, where we also estimate the Boolean complexity of the important problems of multiplication of the transpose of a Vandermonde matrix by a vector and the solution of the transposed Vandermonde linear system of equations.

It is known that the solution of Toeplitz, Hankel, Cauchy, Vandermonde, and transposed Vandermonde linear system of equations is generally prone to numerical stability problems, and numerical problems arise even for multiplication of Cauchy, Vandermonde, and transposed Vandermonde matrices by a vector. Thus our results on the Boolean complexity of these important computations can be quite interesting, because our FFT-based estimates are reasonable, even for more general classes of structured matrices, showing rather moderate growth of the complexity as the input size increases.