Theoretical Computer Science 681 (2017) 217–231

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Computing real witness points of positive dimensional polynomial systems ☆

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ARTICLE INFO

Article history:
Available online 31 March 2017

Keywords:
Numerical algebraic geometry
Real algebraic geometry
Homotopy continuation
Singular critical points
Real witness points

ABSTRACT

We consider a critical point method for finding certain solution (witness) points on real solution components of real polynomial systems of equations. The method finds points that are critical points of the distance from a plane to the component with the requirement that certain regularity conditions are satisfied. In this paper we analyze the numerical stability and complexity of the method. We aim to find at least one well conditioned witness point on each connected component by using perturbation, path tracking and projection techniques. An optimal-direction strategy and an adaptive step size control strategy for path following on high dimensional components are given.

1. Introduction

This paper is a contribution to the development of numerical algorithms for computational algebraic geometry pioneered by Sommese, Wampler, Verschelde and others [34,7]. Recently this direction was extended to the real case by Lu [25], Besana et al. [10], and Hauenstein [18]. Important early work on exactly obtaining real points on components is given by Rouillier et al. [27] and Safey El Din et al. [29,30].

We are aiming to find at least one point in each connected component of a polynomial system. A widely used method is the so-called "critical point" method. In symbolic computation, the study of a smooth and compact positive dimensional real variety by choosing a projection to reduce to zero dimensional critical locus can be found in [2]. It is related to the work by Bank et al. [6,5] on polar varieties of complete intersections. To remove the compactness assumption, the authors introduced a distance function to minimize the distance to a given point in [27,1]. A significant improvement due to Safey El Din and Schost to study such projection functions for non-compact connected components can be found in [29].

Related symbolic approaches for computational real algebraic geometry include Sturm's ancient method for counting real roots of a polynomial and Tarski's real quantifier elimination [35,33] and the work about the number of connected components of a semi-algebraic set [17]. They also include cylindrical algebraic decomposition (CAD) introduced by Collins [13] and improved by Hong [19]. Recent improvement of CAD by using triangular decompositions are given in [12,11] for solving semi-algebraic systems. But the double exponential cost of the CAD algorithm [14] is the main barrier to its application. For more references see [3], and the references in [29,30,5].

☆ The work is partly supported by the projects NSFC 11471307, 11671377 and Key Research Program of Frontier Sciences of CAS QYZDB-SSW-SYS026 and cstc2015jcyjys40001.

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http://dx.doi.org/10.1016/j.tcs.2017.03.035
0304-3975/

In contrast, remarkable developments concerning the computation of real radical of zero dimensional polynomial systems due to Laserre et al. [23] are based on moment matrix and numerical semi-definite programming. Recently such moment matrix completion techniques are explored by Zhi et al. in [26] for finding at least one real root of a given semi-algebraic system. Furthermore, based on critical point techniques and moment matrix completion, they studied the computation of verified real solutions on components of positive dimensional systems in [38].

The methods of numerical algebraic geometry [34] compute approximate complex points on all irreducible solution components of multivariate complex systems of polynomial equations. Such points on components of each possible dimension are obtained by slicing with random planes of equal co-dimension. The points are called *witness points* and are computed with efficient homotopy methods. Finding real solution components is a natural extension with many applications. Naive extension of the complex approach by random slicing to the real case fails, since such random planes may not intersect some (e.g. compact) components.

Consider k polynomials from $\mathbb{R}[x_1, ..., x_n]$

$$f = \{f_1, f_2, ..., f_k\} = 0 \tag{1}$$

satisfying the following regularity assumptions:

A_1: $V_{\mathbb{R}}(f_1, f_2, ..., f_i)$ has dimension $n - i$ for $1 \le i \le k$.
A_2: the ideal $I_i = \langle f_1, f_2, ..., f_i \rangle$ is radical for $1 \le i \le k$.

The solution set or variety of $f = 0$ is:

$$V_{\mathbb{R}}(f_1, ..., f_k) = \{x \in \mathbb{R}^n : f_j(x) = 0,\ 1 \le j \le k\} \tag{2}$$

In [37], we construct the following square system to find real witness points

$$F = \{f, \sum_{i=1}^{k} \lambda_i \nabla f_i - \mathbf{n}\} = 0 \tag{3}$$

and an $n - k - 1$ dimensional system

$$f^{(1)} = \{f, x \cdot \mathbf{n} - 1\} = 0 \tag{4}$$

Here $\mathbf{n}$ is a random vector in $\mathbb{R}^n$ and (3) has $n + k$ equations and $n + k$ unknowns $(x, \lambda) = (x_1, ..., x_n, \lambda_1, ..., \lambda_k)$. For Equation (4), we can construct a square system $F^{(1)}$ and an $n - k - 2$ dimensional system $f^{(2)}$ by the same idea. Thus, altogether there are $n - k + 1$ square systems to be solved. The last one $F^{(n-k-1)} := f^{(n-k)}$ is of the form $\{f, L_1, ..., L_{n-k}\}$ where each L_i is a linear equation in x with random real coefficients. Therefore the real witness set of f can be expressed as:

$$W_{\mathbb{R}}(f) = V_{\mathbb{R}}(F) \cup W_{\mathbb{R}}(f^{(1)}) = V_{\mathbb{R}}(F) \cup \cdots \cup V_{\mathbb{R}}(F^{(n-k)}) \tag{5}$$

Related critical point techniques include point-distance formulations to obtain real points as the critical points of the distance from a component to a random point [27,18].

Previously we introduced a perturbation method with the goal of obtaining a real witness point with high accuracy on each real connected solution component of a system. As explained in our earlier work, singularities inevitably occur and it is important to develop approaches to deal with them. The key in our approach to finding well-conditioned witness points is to track a path of a perturbed system to escape the numerically difficult region near the singularity and then project back to the original real variety.

As a sequel of our previous work [37] we will first explore the complexity of computing real points by using the plane-distance construction. Then we will consider numerical aspects of the methods in this paper with the assumption that a real witness point on a positive dimensional component has already been obtained. The main contributions of this paper are two strategies to ensure the success of this method which were not provided in [37]. One is to use adapted step size to avoid jumping to another component during tracking the path by the prediction-projection technique. Another is to determine the direction by which we can escape the singularity as soon as possible.

Shub and Smale's condition number analysis of nonlinear systems can be used to estimate the size of the convergent region of a root and the minimal distance to other roots [15,9]. Based on it Beltran and Leykin gave a rigorous algorithm for homotopy path-following and its complexity analysis to guarantee convergence and avoid path-jumping [8]. For other important related work on bounds see [16] for root isolation of (sparse) multivariate polynomial systems and see [21] for significant work regarding the minimum distance between two real components.

In this paper, we will introduce an alternative way to estimate the root distance which is one of the key points in our step size control. We assume the convergence of Newton iteration during the projection stage and under this assumption we give a more practical method for avoiding the numerical difficulties. It differs from the analysis of Beltran and Leykin in that a rigorous bound is not given on the step size to ensure convergence while Beltran and Leykin give such a bound for certainty at a higher cost. But our method is guaranteed to detect "path jumping". If jumping or divergence happens, we can decrease the step size and eventually it will work for sufficiently small step size.