Theoretical Computer Science 674 (2017) 1–15

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Automaton semigroups: New constructions results and examples of non-automaton semigroups

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ARTICLE INFO

Article history:
Received 17 April 2016
Received in revised form 15 January 2017
Accepted 1 February 2017
Communicated by D. Perrin

Keywords:
Automaton semigroup
Free product
Wreath product
Constructions
Small extensions

ABSTRACT

This paper studies the class of automaton semigroups from two perspectives: closure under constructions, and examples of semigroups that are not automaton semigroups. We prove that (semigroup) free products of finite semigroups always arise as automaton semigroups, and that the class of automaton monoids is closed under forming wreath products with finite monoids. We also consider closure under certain kinds of Rees matrix constructions, strong semilattices, and small extensions. Finally, we prove that no subsemigroup of $(\mathbb{N}, +)$ arises as an automaton semigroup. (Previously, $(\mathbb{N}, +)$ itself was the unique example of a semigroup having the 'general' properties of automaton semigroups (such as residual finiteness, solvable word problem, etc.) but that was known not to arise as an automaton semigroup.)

1. Introduction

Automaton semigroups (that is, semigroups of endomorphisms of rooted trees generated by the actions of Mealy automata) emerged as a generalisation of automaton groups, which arose from the construction of groups having 'exotic' properties, such as the finitely generated infinite torsion group found by Grigorchuk [10], and later proven to have intermediate growth, again by Grigorchuk [8]. The topic of automaton groups has since developed into a substantial theory; see, for example, Nekrashevych's monograph [18] or one of the surveys by the school led by Bartholdi, Grigorchuk, Nekrashevych, and Šunić [1,2,7].

After the foundational work of Grigorchuk, Nekrashevych & Sushchanskii [9, esp. Sec. 4 & Subsec. 7.2], the theory of automaton semigroups has grown into an active research topic. Broadly speaking, there have been two foci of research. First, the study of decision problems: what can be effectively decided about the semigroup generated by a given automaton?

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1 Much of the research leading to this paper was undertaken during a visit by the first author to the Centro de Matemática e Aplicações, Universidade Nova de Lisboa. We thank the centre and university for their hospitality. This visit was partially supported by the FCT exploratory project IF/01622/2013/CP1161/CT0001 (attached to the second author's fellowship). The first author was also supported by the FCT project CEMAT-CIÊNCIAS (UID/Multi/04621/2013).

2 The second author was supported by an Investigador FCT fellowship (IF/01622/2013/CP1161/CT0001). This work was partially supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the project UID/MAT/00297/2013 (Centro de Matemática e Aplicações).

http://dx.doi.org/10.1016/j.tcs.2017.02.003
0304-3975/

For example, the finiteness and torsion problems are now known to be undecidable for general automaton semigroups [6], but particular special cases are decidable [14,13,16]. Second, the study of the class of automaton semigroups: which semigroups arise and do not arise as automaton semigroups? Two particular aspects of this question are whether the class of automaton semigroups is closed under various semigroup constructions, and giving examples of semigroups that do not arise as automaton semigroups. This paper is concerned with both of these aspects.

For some constructions, such as direct products and adjoining a zero or identity, it is straightforward to prove that the class is closed; see [4, Section 5]. For many other natural constructions, the question of closure remains open. For example, whether automaton semigroups are closed under free products is an open question. (This is related to the problem of showing that all free groups arise as automaton groups; the recent positive answer to this question was the culmination of the work of a series of authors; see [22] and the references therein.) The free product of automaton semigroups is, however, at least very close to being an automaton semigroup: in a previous paper, we showed that $(S \star T)^1$ is always an automaton semigroup if S and T are [3, Theorem 3].

Since closure under free products seemed difficult to settle, the second author asked whether free products of finite semigroups always arise as automaton semigroups [4, Open problem 5.8(1)]. In [3, Conjecture 5], we conjectured that the answer was 'no', and suggested a potential counterexample. However, in this paper we prove that the answer is 'yes': free products of finite semigroups always arise as automaton semigroups (Theorem 2). This parallels the result that (group) free products of finite groups arise as automaton groups [11]. More generally, we show in Theorem 3 that the free product of automaton semigroups each containing an idempotent is always an automaton semigroup.

In our previous paper, we also considered whether a wreath product $S \wr T$, where S is an automaton monoid and T is a finite monoid, was necessarily an automaton monoid. We managed to prove that such a wreath product arises as a *submonoid* of an automaton monoid. In this paper, we obtain a complete answer: all such wreath products arise as automaton monoids (Theorem 5).

We consider whether a Rees matrix semigroup over an automaton semigroup is also an automaton semigroup. We do not have a complete answer, but we prove that this holds under certain restrictions (Proposition 6). This is a step towards classifying completely simple automaton semigroups [4, Open problem 5.8(3)].

We prove that a certain kind of strong semilattice of automaton semigroups is itself an automaton semigroup (Proposition 8). This result is then applied when we turn to the question of whether a small extension of an automaton semigroup is necessarily an automaton semigroup. (Recall that if S is a semigroup and T is a subsemigroup of S with $S \setminus T$ finite, then S is a small extension of T and T is a large subsemigroup of S.) Many finiteness properties are known to be preserved on passing to large subsemigroups and small extensions; see the survey [5]. It is already known that a large subsemigroup of an automaton semigroup is not necessarily an automaton semigroup, for $(\mathbb{N} \cup \{0\}, +)$ is an automaton semigroup but $(\mathbb{N}, +)$ is not; see [4, Section 5]. (This example also shows that adding an identity to a non-automaton semigroup can give an automaton monoid; thus the classes of automaton semigroups and monoids seem to be very different.) We do not have a complete answer to the question of closure under small extensions, but we prove some special cases in Section 7. The importance of these results is that *if* the class of automaton semigroups is not closed under forming small extensions, then we have eliminated several standard constructions as potential sources of counterexamples.

In all of the automaton constructions in this paper, we use alphabets of symbols consisting at least partially of tuples of symbols from the automata for the 'base' semigroups of the construction. This seems to be quite a powerful approach, as it allows the automaton to have access to a lot of information at each transition.

Finally, we present new examples of semigroups that do not arise as automaton semigroups. This is an important advance, because a major difficulty in studying the class of automaton semigroups is that if a semigroup has the properties that automaton semigroups have generally, such as residual finiteness [4, Proposition 3.2], solvable word problem, etc., then there are no general techniques for proving it is not an automaton semigroup. In the pre-existing literature, there is a unique example of a semigroup that has these 'general' automaton semigroup properties but that is known *not* to arise as an automaton semigroup: namely, the free semigroup of rank 1 (or, if one prefers, $(\mathbb{N}, +)$) [4, Proposition 4.3]. We prove that no subsemigroup of this semigroup arises as an automaton semigroup, and indeed that no non-trivial subsemigroup of this semigroup with a zero adjoined arises as an automaton semigroup (Theorem 15). Although our proof is specialised, and thus still leaves open the problem of finding a general technique for proving that a semigroup is not an automaton semigroup, we at least now have a countable, rather than singleton, class of non-automaton semigroups that satisfy the usual general properties of automaton semigroups.

2. Preliminaries

In this section we briefly recall the necessary definitions and concepts required in the rest of the paper. For a fuller introduction to automaton semigroups, see the discussion and examples in [4, Sect. 2].

An *automaton* $\mathcal{A}$ is formally a triple (Q, B, δ), where Q is a finite set of *states*, B is a finite alphabet of *symbols*, and δ is a transformation of the set $Q \times B$. The automaton $\mathcal{A}$ is normally viewed as a directed labelled graph with vertex set Q and an edge from q to r labelled by $x \mid y$ when $(q, x)\delta = (r, y)$: