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Algorithmic complexity of weakly semiregular partitioning and the representation number

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A B S T R A C T

A graph G is *weakly semiregular* if there are two numbers a, b, such that the degree of every vertex is a or b. The *weakly semiregular number* of a graph G, denoted by $wr(G)$, is the minimum number of subsets into which the edge set of G can be partitioned so that the subgraph induced by each subset is a weakly semiregular graph. We present a polynomial time algorithm to determine whether the weakly semiregular number of a given tree is two. On the other hand, we show that determining whether $wr(G) = 2$ for a given bipartite graph G with at most three numbers in its degree set is **NP**-complete. Among other results, for every tree T, we show that $wr(T) \leq 2\log_2 \Delta(T) + \mathcal{O}(1)$, where $\Delta(T)$ denotes the maximum degree of T.

A graph G is a $[d, d+s]$-*graph* if the degree of every vertex of G lies in the interval $[d, d+s]$. A $[d, d+1]$-graph is said to be *semiregular*. The *semiregular number* of a graph G, denoted by $sr(G)$, is the minimum number of subsets into which the edge set of G can be partitioned so that the subgraph induced by each subset is a semiregular graph. We prove that the semiregular number of a tree T is $\lceil \frac{\Delta(T)}{2} \rceil$. On the other hand, we show that determining whether $sr(G) = 2$ for a given bipartite graph G with $\Delta(G) \leq 6$ is **NP**-complete.

In the second part of the work, we consider the representation number. A graph G has a *representation modulo* r if there exists an injective map $\ell : V(G) \to \mathbb{Z}_r$ such that vertices v and u are adjacent if and only if $|\ell(u) - \ell(v)|$ is relatively prime to r. The *representation number*, denoted by $rep(G)$, is the smallest r such that G has a representation modulo r. Narayan and Urick conjectured that the determination of $rep(G)$ for an arbitrary graph G is a difficult problem [38]. In this work, we confirm this conjecture and show that if $\mathbf{NP} \neq \mathbf{P}$, then for any $\epsilon > 0$, there is no polynomial time $(1-\epsilon)\frac{n}{2}$-approximation algorithm for the computation of representation number of regular graphs with n vertices.

1. Introduction

The paper consists of two parts. In the first part, we consider the problem of partitioning the edges of a graph into regular and/or locally irregular subgraphs. In this part, we present some polynomial time algorithms and **NP**-hardness results. In the second part of the work, we focus on the representation number of graphs. It was conjectured that the determination of $rep(G)$ for an arbitrary graph G is a difficult problem [38]. In this part, we confirm this conjecture and

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show that if **NP** $\neq$ **P**, then for any $\epsilon > 0$, there is no polynomial time $(1-\epsilon)\frac{n}{2}$-approximation algorithm for the computation of representation number of regular graphs with n vertices.

2. Partitioning the edges of graphs

In 1981, Holyer in [31] focused on the computational complexity of edge partitioning problems and proved that for each t, $t \geq 3$, it is **NP**-complete to decide whether a given graph can be edge-partitioned into subgraphs isomorphic to the complete graph K_t. Afterwards, the complexity of edge partitioning problems have been studied extensively by several authors, for instance see [21–23]. Nowadays, the computational complexity of edge partitioning problems is a well-studied area of graph theory and computer science. For more information we refer the reader to a survey on graph factors and factorization by Plummer [40].

If we consider the Holyer problem for a family $\mathcal{G}$ of graphs instead of a fixed graph then, we can discover interesting problems. For a family $\mathcal{G}$ of graphs, a $\mathcal{G}$-decomposition of a graph G is a partition of the edge set of G into subgraphs isomorphic to members of $\mathcal{G}$. Problems of $\mathcal{G}$-decomposition of graphs have received a considerable attention, for example, Holyer proved that it is **NP**-hard to edge-partition a graph into the minimum number of complete subgraphs [31]. To see more examples of $\mathcal{G}$-decomposition of graphs see [15,19,33].

2.1. Related works and motivations

We say that a graph is *locally irregular* if its adjacent vertices have distinct degrees and a graph is *regular* if each vertex of the graph has the same degree. In 2001, Kulli et al. introduced an interesting parameter for the partitioning of the edges of a graph [34]. The *regular number* of a graph G, denoted by $reg(G)$, is the minimum number of subsets into which the edge set of G can be partitioned so that the subgraph induced by each subset is regular. The *edge chromatic number* of a graph, denoted by $\chi'(G)$, is the minimum size of a partition of the edge set into 1-regular subgraphs and also, by Vizing's theorem [45], the edge chromatic number of a graph G is equal to either $\Delta(G)$ or $\Delta(G)+1$, therefore the regular number problem is a generalization for the edge chromatic number and we have the following bound: $reg(G) \leq \chi'(G) \leq \Delta(G)+1$. It was asked in [27] to determine whether $reg(G) \leq \Delta(G)$ holds for all connected graphs.

Conjecture 1 *(The degree bound [27]). For any connected graph G, $reg(G) \leq \Delta(G)$.*

It was shown in [4] that not only there exists a counterexample for the above-mentioned bound but also for a given connected graph G decide whether $reg(G) = \Delta(G)+1$ is **NP**-complete. Also, it was shown that the computation of the regular number for a given connected bipartite graph G is **NP**-hard [4]. Furthermore, it was proved that determining whether $reg(G) = 2$ for a given connected 3-colorable graph G is **NP**-complete [4].

On the other hand, Baudon et al. introduced the notion of edge partitioning into locally irregular subgraphs [12]. In this case, we want to partition the edges of the graph G into locally irregular subgraphs, where by a partitioning of the graph G into k locally irregular subgraphs we refer to a partition $E_1, \dots, E_k$ of $E(G)$ such that the graph $G[E_i]$ is locally irregular for every i, $i = 1, \dots, k$. The *irregular chromatic index* of G, denoted by χ'_{irr}, is the minimum number k such that the graph G can be partitioned into k locally irregular subgraphs. Baudon et al. characterized all graphs which cannot be partitioned into locally irregular subgraphs and call them exceptions [12]. Motivated by the 1–2–3-Conjecture, they conjectured that apart from these exceptions all other connected graphs can be partitioned into three locally irregular subgraphs [12]. For more information about the 1–2–3-Conjecture and its variations, we refer the reader to a survey on the 1–2–3 Conjecture and related problems by Seamone [43] (see also [1,2,11,20,14,42,44]).

Conjecture 2 *([12]). For every non-exception graph G, we have $\chi'_{irr}(G) \leq 3$.*

Regarding the above-mentioned conjecture, Bensmail et al. in [16] proved that every bipartite graph G which is not an odd length path satisfies $\chi'_{irr}(G) \leq 10$. Also, they proved that if G admits a partitioning into locally irregular subgraphs, then $\chi'_{irr}(G) \leq 328$. Recently, Lužar et al. improved the previous bound for bipartite graphs and general graphs to 7 and 220, respectively [36]. For more information about this conjecture see [41].

Regarding the complexity of edge partitioning into locally irregular subgraphs, Baudon et al. in [13] proved that the problem of determining the irregular chromatic index of a graph can be handled in linear time when restricted to trees. Furthermore, in [13], Baudon et al. proved that determining whether a given planar graph G can be partitioned into two locally irregular subgraphs is **NP**-complete.

In 2015, Bensmail and Stevens considered the problem of partitioning the edges of graph into some subgraphs, such that in each subgraph every component is either regular or locally irregular [18]. The *regular-irregular chromatic index* of graph G, denoted by $\chi'_{reg-irr}$, is the minimum number k such that G can be partitioned into k subgraphs, such that each component of every subgraph is locally irregular or regular [18]. They conjectured that the edges of every graph can be partitioned into at most two subgraphs, such that each component of every subgraph is regular or locally irregular [17,18].

Conjecture 3 *([17,18]). For every graph G, we have $\chi'_{reg-irr}(G) \leq 2$.*