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The within-strip discrete unit disk cover problem ☆,☆☆

Robert Fraser *, Alejandro López-Ortiz

School of Computer Science, University of Waterloo, Waterloo, Ontario, Canada N2L 3G1

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A B S T R A C T

We present a study of the Within-Strip Discrete Unit Disk Cover (WSDUDC) problem, which is a restricted version of the Discrete Unit Disk Cover (DUDC) problem. For the WSDUDC problem, there exist a set of points and a set of unit disks in the plane, and the points and disk centres are confined to a strip of fixed width. An optimal solution to the WSDUDC problem is a subset of the disks of minimum cardinality that covers all points in the input set. We describe two approximation algorithms for the problem: a 3-approximate algorithm which applies for strips of width at most 0.8 units, and a general scheme for any strip with less than unit width. We prove that the WSDUDC problem is NP-hard on strips of any fixed width, which is our most interesting result from a theoretical standpoint. The result is also quite surprising, since a number of similar problems are tractable on strips of fixed width. Finally, we discuss how these results may be applied to known DUDC approximation algorithms.

1. Introduction

We consider the Within-Strip Discrete Unit Disk Cover (WSDUDC) problem, which is a special case of the general Discrete Unit Disk Cover (DUDC) problem. Given a set $\mathcal{D}$ of m unit disks and a set $\mathcal{P}$ of n points in the plane, the DUDC problem is to select a minimum cardinality subset $\mathcal{D}^\star \subseteq \mathcal{D}$ to cover $\mathcal{P}$. Let a *strip* of the plane be the locus of the points between two horizontal lines. If both $\mathcal{P}$ and the centrepoints of $\mathcal{D}$ are located within some strip of the plane of fixed width, then the problem is an instance of the WSDUDC problem.

We show that the WSDUDC problem is NP-hard for strips of any fixed width. The hardness result is surprising because there are a number of very similar problems that are tractable if the width of the strip is bounded by a constant, such as finding the maximum independent set of a unit disk graph [2]. The Geometric Set Cover problem on unit squares (precisely the WSDUDC problem, except the disks are replaced with axis-aligned unit squares) can be solved optimally in $n^{O(k)}$-time when confined to strips of width k [3]. The problem of covering points in a strip with disks centred strictly outside the strip (rather than inside as in the WSDUDC problem) is polynomial time solvable [4]. The notion of decomposing a problem into strip-based subproblems is a standard technique [5]. An implication of a polynomial time algorithm for the WSDUDC problem for strips of any fixed width is a simple PTAS for the DUDC problem using the shifting techniques of Hochbaum and Maass [5]. The PTAS for the DUDC problem [6], discussed in Section 2.1, uses fundamentally different techniques.

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* Corresponding author.
E-mail addresses: fraser@cs.umanitoba.ca (R. Fraser), alopez-o@uwaterloo.ca (A. López-Ortiz).

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The formal definition of the WSDUDC problem is provided in Definition 1. For $i \in \{1, 2\}$, let ℓ_i^+ denote the open half-plane lying above the horizontal line ℓ_i, and similarly ℓ_i^- is the open half-plane lying below ℓ_i. We say that a disk of radius r centred at a point q *covers* a point p if the Euclidean distance between p and q is no greater than r. Similarly, a set of disks $\mathcal{D}'$ covers a set of points $\mathcal{P}'$ if $\mathcal{P}' \subset \bigcup_{D\in\mathcal{D}'} D$. In this work, all disks have unit radius.

Definition 1. *The Within-Strip Discrete Unit Disk Cover (WSDUDC) Problem:* The input to the problem consists of a set $\mathcal{P}$ of n points, a set $\mathcal{D}$ of m unit radius disks whose centrepoints define the set $\mathcal{Q}$, and the strip $s = \ell_1^- \cap \ell_2^+$ of width h defined by parallel lines ℓ_1 and ℓ_2 (all in the Euclidean plane), where $\ell_1 \subset \ell_2^+$, $\ell_2 \subset \ell_1^-$, $\mathcal{P} \subset s$ and $\mathcal{Q} \subset s$. The Within-Strip Discrete Unit Disk Cover problem is to find a subset $\mathcal{D}^\star \subseteq \mathcal{D}$ of minimum cardinality so that $\mathcal{P} \subset \bigcup_{D\in\mathcal{D}^\star} D$.

We assume that we are provided with the lines ℓ_1 and ℓ_2 as part of the input; alternatively, another version of the problem may require that a minimum width strip is computed (e.g. [7]). For our discussion, we assume that the provided strip is horizontal. Let *left* be defined as one of the horizontal directions (which direction is chosen must be done so consistently), and let *right* be the opposite direction.

This is a seemingly simpler context than the general Discrete Unit Disk Cover (DUDC) problem, which is only distinguished from the WSDUDC problem by the fact that the DUDC problem has no strip confining the positions of the points and disks.

Definition 2. *The Discrete Unit Disk Cover (DUDC) Problem:* Given a set $\mathcal{P}$ of n points and a set $\mathcal{D}$ of m unit radius disks in the plane, the Discrete Unit Disk Cover problem is to find a subset $\mathcal{D}^\star \subseteq \mathcal{D}$ of minimum cardinality such that $\mathcal{P} \subset \bigcup_{D\in\mathcal{D}^\star} D$.

The DUDC problem is NP-hard[1] [8,9], and the general set cover problem (i.e. the covering sets are unrestricted) is not approximable within a factor of $c \log n$, for some constant c, assuming $\mathsf{P} \neq \mathsf{NP}$ [10].

1.1. Applications

While we do not claim any specific applications for this research, work on specific geometric set cover problems is often motivated by applications in wireless networking or facility location problems, e.g. [11,12]. In particular, when wireless clients and servers are modelled as points in the plane and the range of wireless transmission is assumed to be constant in all directions (say one unit), the resulting coverage region is a disk of unit radius centred on the point representing the wireless transmitting device. Under this model, a sender a successfully transmits a wireless message to a receiver b if and only if the point b is covered by the unit disk centred at the point a. This model applies more generally to a variety of facility location problems for which the Euclidean distance between clients and facilities cannot exceed a given radius, and clients and candidate facility locations are represented by discrete sets of points. Problems of this nature may be modelled with the DUDC problem.

Our interest is primarily theoretical. However, since the WSDUDC problem is an important subroutine for several DUDC approximation algorithms, improvements to this problem address the general DUDC problem as well.

1.2. Organization of the paper

We begin with a review of the general Discrete Unit Disk Cover (DUDC) problem and several related problems and restricted settings of the DUDC problem. We describe two approximation algorithms for the WSDUDC problem, and we provide a proof demonstrating that the WSDUDC problem is NP-hard on strips of any fixed width. The approximation algorithms for the problem are:

- a general $(2\lceil 1/(2\sqrt{1-h^2}) - 1/2\rceil + 2\lceil 1/\sqrt{1-h^2}\rceil + 2)$-approximation algorithm that runs in $O(m^4 n + n \log n)$-time on strips of width $h < 1$, and
- a 3-approximation algorithm running in $O(m^6 n + n \log n)$-time on strips of width $h \leq 4/5$.

Finally, we describe how these algorithms may be used to obtain new approximation algorithms for the general DUDC problem.

2. Related work

There are many known approximation algorithms for the DUDC problem, which make use of a variety of techniques. The general approximation algorithms for the Set Cover problem (e.g. [13]) directly provide an $O(\log n)$-approximation algorithm. There is a series of constant factor approximation algorithms and a PTAS for the DUDC problem, mostly published within

[1] To be precise, it is NP-hard in dimensions $d > 1$, but is solvable in linear time in $d = 1$ [5].

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