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Edge motion and the distinguishing index ☆

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The *distinguishing index* $D'(G)$ of a graph G is the least number d such that G has an edge colouring with d colours that is preserved only by the trivial automorphism. We investigate the edge motion of a graph with respect to its automorphisms and compare it with the vertex motion. We prove an analog of the Motion Lemma of Russell and Sundaram, and we use it to determine the distinguishing index of powers of complete graphs and of cycles with respect to the Cartesian, direct and strong product.

1. Introduction

The *distinguishing index* $D'(G)$ of a graph G is the least number d such that G has an edge colouring with d colours that is preserved only by the trivial automorphism. This notion was introduced by Kalinowski and Pilśniak [10] as an analog of the well-known *distinguishing number* $D(G)$ of a graph G defined by Albertson and Collins [2] for vertex colourings. Symmetry breaking (in various ways) has interesting applications to numerous problems of theoretical computer science, for instance to the leader election problem and self-stabilizing algorithms (cf. [5,7,9]).

Obviously, the distinguishing index is not defined for K_2, thus from now on, we assume that K_2 is not a connected component of any graph being considered. There are graphs G with $D'(G) = D(G)$. Easy examples are paths and cycles: $D'(P_n) = D'(C_p) = 2$, for any $n \geq 3$ and any $p \geq 6$, and $D'(C_3) = D'(C_4) = D'(C_5) = 3$. It is also possible that $D'(G) > D(G)$, and a class of trees satisfying this inequality was found in [10]. However, very often $D'(G) < D(G)$. For example, $D'(K_n) = D'(K_{p,p}) = 2$, for any $n \geq 7$ and for any $p \geq 4$ (see [10]) while $D(K_n) = n$ and $D(K_{p,p}) = p + 1$.

A general sharp upper bound for $D'(G)$ was proved in [10].

Theorem 1. [10] *If G is a finite connected graph of order $n \geq 3$, then*

$$D'(G) \leq D(G) + 1.$$

Moreover, if $\Delta(G)$ is the maximum degree of G, then

$$D'(G) \leq \Delta(G)$$

unless G is a C_3, C_4 or C_5.

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In [10], all trees T with $D'(T) = \Delta(T)$ were characterized.

Given an automorphism φ of a graph $G = (V, E)$, let V_φ denote the set of all vertices φ moves:

$$V_\varphi = \{v \in V : \varphi(v) \neq v\}.$$

The *motion of an automorphism* φ of a graph G is the number $m(\varphi) = |V_\varphi|$, and the *motion of a graph* G is defined as

$$m(G) = \min\{m(\varphi) : \varphi \in \mathrm{Aut}(G) \setminus \{\mathrm{id}\}\}.$$

Russell and Sundaram [12] proved that the distinguishing number of a graph is small when every (nontrivial) automorphism of G moves many vertices. The precise statement of this powerful result follows.

Theorem 2 *(Russell–Sundaram Motion Lemma [12]). For any graph G and any positive integer d the inequality*

$$d^{\frac{m(G)}{2}} \geq |\mathrm{Aut}(G)|$$

implies $D(G) \leq d$.

In this paper we investigate the edge motion of finite graphs and prove an analogous result. In Section 2 we discuss a relationship between the vertex motion and the edge one of a connected graph. In particular, we obtain an interesting comparison of these two invariants for trees.

In Section 3 we prove the analog of the Motion Lemma of Russell and Sundaram for the motion of edges. We adopt the method of proof of the Motion Lemma [12], but we include this short proof for the sake of completeness. We observe that all graphs with minimum degree at least three which satisfy the hypothesis of the Motion Lemma of Russell and Sundaram for a certain d also satisfy the hypothesis of the Edge Motion Lemma with the same d, by Theorem 4 in Section 2. In such cases, we can similarly infer that $D'(G) \leq d$. But there exist graphs satisfying the hypothesis of the Edge Motion Lemma only (e.g., the Cartesian square of a complete graph of order $n \geq 4$, as we show in the next section). Therefore, in Section 4 we consider mainly such graphs, for which the distinguishing number cannot be determined by use the Motion Lemma. Thus we show how to apply the Edge Motion Lemma to determine the distinguishing index of powers of complete graphs and of cycles with respect to three standard graph products: the Cartesian, direct and strong ones.

The distinguishing index and the edge motion for certain infinite graphs has been investigated in [3].

2. Edge motion compared with vertex motion

Every automorphism $\varphi : V \to V$ of a graph $G = (V, E)$ induces a permutation $\varphi^* : E \to E$ defined as $\varphi^*(uv) = \varphi(u)\varphi(v)$ for every edge $uv \in E$. Let E_φ be the set of edges φ^* moves, i.e.,

$$E_\varphi = \{e \in E : \varphi^*(e) \neq e\}.$$

The number $m^*(\varphi) = |E_\varphi|$ is called the *edge motion of an automorphism* φ, and the *edge motion of a graph* G is defined as

$$m^*(G) = \min\{m^*(\varphi) : \varphi \in \mathrm{Aut}(G) \setminus \{\mathrm{id}\}\}.$$

We set $m^*(G) = 0$ when $\mathrm{Aut}(G) = \{\mathrm{id}\}$. For example, $m^*(K_n) = 2n - 4$ while $m(K_n) = 2$, $m^*(C_{2k}) = m(C_{2k}) = 2k - 2$, $m^*(P_{2k+1}) = m(P_{2k+1}) = 2k$, but $m^*(P_{2k}) = 2k - 2 = m(P_{2k}) - 2$.

Let us compare the motion and the edge motion of graphs in general. For trees, we have the following relationship which is fully covered by paths depending on their parity. Let us recall the well-known fact that every tree has either a central vertex or a central edge, and it is fixed by every of its automorphisms.

Proposition 3. *If T is a tree, then either $m^*(T) = m(T)$ or $m^*(T) = m(T) - 2$. Moreover, the latter case holds if and only if $\mathrm{Aut}(T) \neq \{\mathrm{id}\}$ and T has a central edge e_0 and every non-trivial automorphism of T switches the end vertices of e_0.*

Proof. First assume that T has a central vertex v_0. To prove that $m^*(T) = m(T)$ in this case it suffices to show that for any $\varphi \in \mathrm{Aut}(T)$ there is a bijection between the set V_φ of vertices and the set E_φ of edges that are moved by φ. For any vertex $v \neq v_0$, let v^- denote its neighbour situated on the path between v and v_0. Clearly, $vv^- \in E_\varphi$ if and only if $v \in V_\varphi$. As v_0 is fixed by φ, the correspondence

$$V_\varphi \ni v \mapsto vv^- \in E_\varphi$$

is one-to-one.

Now, let T contain a central edge $e_0 = u_0v_0$. If the vertices u_0, v_0 are fixed by a certain nontrivial automorphism φ, then $m^*(\varphi) = m(\varphi)$, and consequently $m^*(T) = m(T)$ due to the same arguments as in the case of a central vertex. If $\varphi(u_0) = v_0$, and thus $\varphi(v_0) = u_0$, for every $\varphi \in \mathrm{Aut}(T) \setminus \{\mathrm{id}\}$, then $V_\varphi = V$ and $E_\varphi = E \setminus \{e_0\}$. Consequently, $m^*(T) = m(T) - 2$. $\square$