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Ordinal proximity measures in the context of unbalanced qualitative scales and some applications to consensus and clustering

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ABSTRACT

In this paper, we introduce ordinal proximity measures in the setting of unbalanced qualitative scales by comparing the proximities between linguistic terms without numbers, in a purely ordinal approach. With this new tool, we propose how to measure the consensus in a set of agents when they assess a set of alternatives through an unbalanced qualitative scale. We also introduce an agglomerative hierarchical clustering procedure based on these consensus measures.

1. Introduction

In different decision-making problems, agents have to show their opinions on a set of alternatives and then an aggregation procedure is used for generating a collective outcome: a winning alternative, several winning alternatives, a ranking on the set of alternatives, etc.

The agents opinions can be provided in very different ways: the favorite alternative, a subset of acceptable alternatives, a ranking on the set of alternatives, an assessment for each alternative, etc.

When agents assess independently each alternative, the corresponding assessments can be of different nature depending on the context: numerical values, intervals of real numbers, fuzzy numbers, linguistic terms, etc.

Qualitative scales are formed by linguistic terms. Usually, these scales are balanced and uniform: there are the same number of positive and negative terms, and adjacent terms are equidistant (for instance, 'very bad', 'bad', 'acceptable', 'good' and 'very good'). However, sometimes the qualitative scales are unbalanced: there are different number of positive terms compared to negative ones,[1] and it is not clear how to measure the nearness between the linguistic terms of an unbalanced qualitative scale.[2]

In this paper, we do not assign numerical distances between linguistic terms, but we propose to make pairwise comparisons of psychological proximities between them. This approach has some similarities with *difference measurement* within the classical measurement theory (see Krantz et al. [32, chapter 4] and Roberts [40, section 3.3]), and also with *non-metric multidimensional scaling*, where only the ranks of the psychological distances or proximities are known (see Bennett and Hays [5], Shepard [41], Coombs [12], Kruskal and Wish [33], Cox and Cox [13] and Borg and Groenen [7, chapter 9], among others). We have also to mention Bossert et al. [9] that consider ordinal measures of distances in the analysis of diversity.

In order to explain how the mentioned comparisons can be made, we consider, as an example, that some journals use the linguistic terms 'reject', 'major revision', 'minor revision' and 'accept' in the evaluation of papers. It has no sense to assign numerical values neither to these terms nor to distances between terms. However, an author may feel that the psychological proximity between 'minor revision' and 'accept' is bigger than the

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[1] For instance, Herrera et al. [24] consider the following nine linguistic terms: 'none', 'low', 'medium', 'almost high', 'high', 'quite high', 'very high' and 'total'.

[2] Nevertheless, within a fuzzy approach, some cardinal proposals on unbalanced qualitative scales can be found in Herrera et al. [24] and Cabrerizo et al. [11], among others.

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psychological proximity between 'minor revision' and 'major revision'. Obviously, this author could compare psychological proximities between the rest of pairs of linguistic terms. Initially, this task may seem hard, because there are $16^2 = 256$ possible pairwise comparisons. Fortunately, it is not necessary to compare all the pairs[3]: the psychological proximity between two terms does not depend on the order these terms are presented; the psychological proximity between a term and itself is always the same and it is bigger than the psychological proximity between two different terms; etc.

Taking into account the previous ideas, we propose the notion of *ordinal proximity measure* as a mapping that assigns an element of a chain to each pair of psychological proximities between linguistic terms, satisfying four independent properties: all the elements in the chain correspond to some psychological proximity, i.e., no element in the chain is superfluous; psychological proximities are symmetric, i.e., the order of the pairs is irrelevant in the comparison; the maximum psychological proximity is reached when comparing a linguistic term with itself; and given three different linguistic terms, the degree of proximity between the lowest and the highest terms should be smaller than the degrees of proximity between the lowest and the intermediate terms and between the intermediate and the highest terms.

Once the ordinal proximity measuring model has been introduced, we propose consensus measures and agglomerative hierarchical clustering procedures when a group of agents evaluate the alternatives through a qualitative scale, taking into account the ordinal proximities between individual assessments.

Given a subset of agents and a subset of alternatives, we define the degree of consensus as the upper median of the proximities between all the pairs of individual assessments. We propose a sequential tie-breaking process and provide some properties of the degrees of consensus.

We have also devised an agglomerative hierarchical clustering procedure where agents are grouped into clusters by defining the similarity between two groups of agents with respect to a subset of alternatives as the degree of consensus in the merged group. We have illustrated our proposal from the qualitative marks obtained by a group of students in several subjects.

The rest of the paper is organized as follows. Section 2 is devoted to introduce and analyze ordinal proximity measures. In Section 3 we propose some applications to consensus and clustering. And Section 4 includes some concluding remarks.

2. Ordinal proximity measures

Let $A = \{1, \ldots, m\}$, with $m \geq 2$, be a set of agents and let $X = \{x_1, \ldots, x_n\}$, with $n \geq 2$, be the set of alternatives which have to be evaluated. Each agent assigns a linguistic term to every alternative within a finite linguistic ordered scale $\mathcal{L} = \{l_1, \ldots, l_g\}$, arranged from the lowest to the highest terms,[4] where the granularity of $\mathcal{L}$ is at least 3 ($g \geq 3$).

2.1. The model

Consider that the psychological proximity between $l_r \in \mathcal{L}$ and $l_s \in \mathcal{L}$ is represented by π_{rs} and let $\Delta = \{\pi_{rs} \mid r, s \in \{1, \ldots, g\}\}$ be the set of all possible psychological proximities between linguistic terms.[5]

Although we do not associate numbers to psychological proximities, we assume that it is possible to compare psychological proximities between linguistic terms through an asymmetric and transitive binary relation $\succ$ on Δ, where $\pi_{rs} \succ \pi_{tu}$ means that the psychological proximity between l_r and l_s is bigger than the psychological proximity between l_t and l_u.

We consider that the following properties should be satisfied for all $r, s, t, u \in \{1, \ldots, g\}$:

1. If neither $\pi_{rs} \succ \pi_{tu}$ nor $\pi_{tu} \succ \pi_{rs}$, then $\pi_{rs} = \pi_{tu}$.
2. $\pi_{sr} = \pi_{rs}$.
3. $\pi_{rr} = \pi_{ss}$.
4. If $s \neq t$, then $\pi_{rr} \succ \pi_{st}$.
5. If $r < s < t$, then $\pi_{rs} \succ \pi_{rt}$ and $\pi_{st} \succ \pi_{rt}$.
6. If $r < s$ and $(r, s) \neq (1, g)$, then $\pi_{rs} \succ \pi_{1g}$.

We now introduce a formal notion of proximity between linguistic terms with values on a finite chain (linear order) $\Delta = \{\delta_1, \ldots, \delta_h\}$, with $\delta_1 \succ \cdots \succ \delta_h$, that captures the properties introduced above. The elements of Δ have no meaning and they only represent different degrees or proximity, being δ_1 and δ_h the maximum and minimum degrees of proximity, respectively.

As usual in the setting of linear orders, $\delta_r \prec \delta_s$ means $\delta_s \succ \delta_r$; $\delta_r \preceq \delta_s$ means $\delta_r \prec \delta_s$ or $\delta_r = \delta_s$; and $\delta_r \succeq \delta_s$ means $\delta_r \succ \delta_s$ or $\delta_r = \delta_s$.

First we assume that all the elements of Δ are relevant because they are reached as the degree of proximity between at least a pair of linguistic terms (exhaustiveness). We also assume that the proximity between a pair of linguistic terms does not depend on the order these terms are presented (symmetry), and the maximum proximity between linguistic terms is only reached when comparing a term with itself. Additionally, we assume that, given three different linguistic terms, the degree of proximity between the lowest and the highest terms should be smaller than the degrees of proximity between the lowest and the intermediate terms and also between the intermediate and the highest terms (monotonicity).

Definition 1. An *ordinal proximity measure on $\mathcal{L}$ with values in Δ* is a mapping $\pi : \mathcal{L}^2 \longrightarrow \Delta$, where $\pi(l_r, l_s) = \pi_{rs}$ means the degree of proximity between l_r and l_s, satisfying the following conditions:

1. *Exhaustiveness*: For every $\delta \in \Delta$, there exist $l_r, l_s \in \mathcal{L}$ such that $\delta = \pi_{rs}$.
2. *Symmetry*: $\pi_{sr} = \pi_{rs}$, for all $r, s \in \{1, \ldots, g\}$.
3. *Maximum proximity*: $\pi_{rs} = \delta_1 \Leftrightarrow r = s$, for all $r, s \in \{1, \ldots, g\}$.
4. *Monotonicity*: $\min\{\pi_{rs}, \pi_{st}\} \succ \pi_{rt}$, for all $r, s, t \in \{1, \ldots, g\}$ such that $r < s < t$.

Every ordinal proximity measure can be represented by a $g \times g$ symmetric matrix with coefficients in Δ, where the elements in the main diagonal are $\pi_{rr} = \delta_1$, $r = 1, \ldots, g$:

$$\begin{pmatrix} \pi_{11} & \cdots & \pi_{1s} & \cdots & \pi_{1g} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \pi_{r1} & \cdots & \pi_{rs} & \cdots & \pi_{rg} \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \pi_{g1} & \cdots & \pi_{gs} & \cdots & \pi_{gg} \end{pmatrix} = (\pi_{rs}).$$

This matrix is called *proximity matrix*.

[3] In Remark 1 we show that, with four linguistic terms, only between three and six comparisons are needed.

[4] For instance, Balinski and Laraki [4] consider the following six linguistic terms: 'to reject' (l_1), 'poor' (l_2), 'acceptable' (l_3), 'good' (l_4), 'very good' (l_5) and 'excellent' (l_6).

[5] At this stage we do not specify what kind of mathematical objects represent psychological proximities.