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Kobayashi compressibility ☆

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ABSTRACT

Kobayashi [21] introduced a uniform notion of compressibility of infinite binary sequences X in terms of relative Turing computations with sub-identity use of the oracle. Given $f : \mathbb{N} \to \mathbb{N}$ we say that X is f-compressible if there exists Y such that for each n we compute $X\upharpoonright_n$ using at most the first $f(n)$ bits of the oracle Y. Kobayashi compressibility has remained a relatively obscure notion, with the exception of some work on resource bounded Kolmogorov complexity. The main goal of this note is to show that it is relevant to a number of topics in current research on algorithmic randomness.
We prove that Kobayashi compressibility can be used in order to define Martin-Löf randomness, a strong version of finite randomness and Kurtz randomness, strictly in terms of Turing reductions. Moreover these randomness notions naturally correspond to Turing reducibility, weak truth-table reducibility and truth-table reducibility respectively. Finally we discuss Kobayashi's main result from [21] regarding the compressibility of computably enumerable sets, and provide additional related original results.

1. Introduction

The compressibility of a finite binary program σ is defined in terms of the shortest program that can generate σ. This is the idea behind the theory of Kolmogorov complexity C of strings. For example, if $c \in \mathbb{N}$ then σ is c-incompressible if $C(\sigma) \geq |\sigma| - c$, and similar definitions are used with respect to the prefix-free complexity K, where the underlying universal machine is prefix-free. This notion of incompressibility has a well-known extension to infinite binary streams X, where we say that X is c-incompressible if $K(X\upharpoonright_n) \geq n - c$ for all n. Then the algorithmic randomness of X is often identified with the property that X is c-incompressible for some c, and coincides with the notion of Martin-Löf randomness.[1] These concepts are basic in Kolmogorov complexity, and the reader is referred to the standard textbooks [29,15] for the relevant background.

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[1] This is the first robust and most accepted definition of algorithmic randomness and was originally introduced by Martin-Löf [31] based on effective statistical tests.

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1.1. Kobayashi compressibility and incompressibility

The reader may observe that the above extension of the definition of compressibility from finite to infinite sequences is nonuniform, in the sense that the compression of the various initial segments of X could be done by different, possibly unrelated short programs. A uniform extension of compressibility from strings to infinite streams would require the individual short programs to be part of a single stream. Kobayashi [21] considered exactly that approach.

Definition 1.1 *(Kobayashi [21]).* Given $f:\mathbb{N}\to\mathbb{N}$ we say that X is f-compressible if there exists Y which computes X via an oracle Turing machine which queries, for each n, at most the first $f(n)$ digits of Y for the computation of $X\upharpoonright_n$.[2]

Note that, since every real is computable from itself with identity use, Definition 1.1 only makes sense if $f(n)$ occasionally dips well below n. This feature contrasts a standard caveat that is often assumed in computability theory for convenience, that the oracle-use in relative computations is strictly increasing. Kobayashi did not necessarily require that f is computable in this definition, but added effectivity requirements in the statements of his results. We formulate the corresponding notion of incompressibility of a real X based on Definition 1.1 as follows.

Definition 1.2 *(Kobayashi incompressibility).* We say that a real X is Kobayashi incompressible if it is not f-compressible for any function f such that $n-f(n)$ is unbounded.

Note that every set X is $(n-c)$-compressible for every constant c. Indeed, given c one can consider Y such that $X = X\upharpoonright_c * Y$, and by hardwiring $X\upharpoonright_c$ into a Turing machine we can compute X from Y with oracle-use $n-c$.

Kobayashi showed that the class of the incompressible streams X of Definition 1.2 has measure 1. We will see in the following that, in fact, this definition is equivalent to Martin-Löf randomness. Furthermore, if Turing computability in this definition is replaced with stronger reducibilities, then we get alternative definitions of Kurtz randomness[3] and a strong version of computably bounded randomness[4] which we call granular randomness, respectively. We state these results in Section 1.4, deferring their proofs in latter sections. It is interesting to note that these alternative definitions do not involve measure or prefix-free machines, so they are unique in that they only use notions from classical computability theory. It is curious that Kobayashi's simple and natural notion of compressibility has remained rather obscure, and does not even feature in the encyclopedic books on Kolmogorov complexity and computability [29,15,32].[5]

1.2. Oracle-use in computations

Note that if f is non-computable, then the condition in Definition 1.1 does not necessarily mean that X is computable from Y with oracle use f. The results we present often hide a non-standard notion of oracle-use in computations, and for this reason we introduce some basic terminology. We define *oracle-use* in a computation of X from Y through an oracle Turing machine in the standard way, as the function $n\mapsto f(n)$ which indicates, for each n, the largest position in Y which was queried during the computation of $X(n)$. Note that this oracle use is computable in the oracle Y (but in general non-computable), and it is *adaptive*, in the sense that it depends on the oracle Y. Another standard notion is the oracle-use of a truth-table reduction $X\le_{tt} Y$. In this case the *oracle-use of the truth-table reduction* is the function $n\mapsto g(n)$ which indicates, for each n, the largest position in Y which occurs in the truth-table corresponding to the computation of $X(n)$. Note that the oracle-use of a truth-table reduction is computable and *oblivious* in the sense that it does not depend on the oracle Y. Finally a weak-truth-table reduction $X\le_{wtt} Y$ is exhibited by a Turing machine $M(n)$ and a computable function h such that the oracle-use of $M^Z(n)$ is bounded above by $h(n)$ for all oracles Z and all numbers n such that $M^Z(n)\downarrow$. In this case h is called the oracle-use of the weak-truth-table reduction, and it is *oblivious* and computable by definition.

We now introduce a non-standard definition. Day [13] used the following notion in order to characterize various notions of algorithmic randomness (see Section 1.3). We say that X is totally Turing reducible to Y with oracle-use f is there is a total Turing machine M which computes X with oracle Y and oracle-use f. Recall that $X\le_{tt} Y$ if and only if there is a total Turing machine M (i.e. such that $n\mapsto M^Z(n)$ is total for all Z) which computes X with oracle Y. The truth-table oracle-use is oblivious and computable while the oracle-use of total reductions is adaptive and could be incomputable. However the oracle-use of a total reduction has a computable upper bound, and it is computable in the oracle Y. Day [13] provided characterizations of various notions of algorithmic randomness based on the oracle-use in total reductions. We briefly discuss these contributions in Section 1.3, in the context of the present paper.

[2] The reader who is familiar with monotone complexity from Levin in [26,27] (also discussed in [15, Section 3.15]) will note that if X is f-compressible for a computable function f, then f is an upper bound on the monotone complexity of X.

[3] Originally from Kurtz [25] and further studied by Wang [36] and Downey Griffiths and Reid [14].

[4] Introduced and studied by Brodhead, Downey and Ng [10].

[5] Of the two citations to Kobayashi's work in [29] one is about a somewhat known result regarding the structure of one-tape nondeterministic Turing machine time hierarchy and the other is [22]. Incidentally, the results in the latter paper were independently reproved by Becher, Figueira, Grigorieff and Miller [9] (along with other original results).

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