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Exact exponential algorithms to find tropical connected sets of minimum size ☆

Mathieu Chapelle [a], Manfred Cochefert [b], Dieter Kratsch [b,*], Romain Letourneur [c], Mathieu Liedloff [c]

[a] *Université Libre de Bruxelles, CP 212, 1050 Bruxelles, Belgium*
[b] *Université de Lorraine, LITA, 57045 Metz cedex 01, France*
[c] *Université d'Orléans, INSA Centre Val de Loire, LIFO, 45067 Orléans, France*

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Tropical Connected Set is strongly related to the *Graph Motif* problem which deals with vertex-colored graphs. *Graph Motif* has various applications in biology and metabolic networks, and has widely been studied in the last twenty years.

The input of the *Tropical Connected Set* problem is a vertex-colored graph (G, c), where $G = (V, E)$ is a graph and c is a vertex coloring assigning to each vertex of G a color. The task is to find a connected subset $S \subseteq V$ of minimum size such that each color of G appears in S. This problem is known to be NP-complete, even when restricted to trees of height at most three. We study exact exponential algorithms to solve *Tropical Connected Set*. We present an $\mathcal{O}^*(1.5359^n)$ time algorithm for general graphs and an $\mathcal{O}^*(1.2721^n)$ time algorithm for trees. We also show that *Tropical Connected Set* on trees has no sub-exponential algorithm unless the Exponential Time Hypothesis fails.

1. Introduction

Problems on vertex-colored graphs have been widely studied in the last 20 years, notably the *Graph Motif* problem which was introduced in 1994 by McMorris et al. [21]. This problem is motivated by applications in biology and metabolic networks [20,24]. *Graph Motif* is a decision problem asking whether a given vertex-colored graph (G, c) has a connected subset S of vertices such that there is bijection between S and a multiset of colors; the latter being part of the input. Equivalently, the question is whether a vertex-colored graph given with a vector of multiplicities of the colors of G has a connected vertex set S such that each color appears in S with its required multiplicity. As an immediate consequence, the size of a solution S is given as part of the input.

Fellows et al. proved that *Graph Motif* is NP-complete, even if the multiset of colors is actually a set and if the graph is a tree of maximum degree three [13]. (The special case where the multiset of colors is actually a set has also been called the *Colorful Motif* problem [2].) Fellows et al. also proved that *Graph Motif* is NP-complete even if the multiset contains only two colors and if the graph is bipartite of maximum degree four [13]. Many different variants of the *Graph Motif* problem have been studied; typical contributions being NP-hardness proofs and fixed-parameter tractable algorithms [5,6,8,12–14,17,19,

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* Corresponding author.
E-mail addresses: Mathieu.ch@gmail.com (M. Chapelle), manfred.cochefert@univ-lorraine.fr (M. Cochefert), dieter.kratsch@univ-lorraine.fr (D. Kratsch), romain.letourneur@univ-orleans.fr (R. Letourneur), mathieu.liedloff@univ-orleans.fr (M. Liedloff).

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20]. To the best of our knowledge, the unique paper to study exact exponential time algorithms within a variant of *Graph Motif* is the one by Dondi et al. [11]. Another well-studied NP-hard problem typically considered for vertex-colored trees is the optimization problem *Convex Recoloring* which is also motivated by applications in biology [4,7,22].

In this paper we study the optimization problem *Tropical Connected Set* defined as follows. Let $G = (V, E)$ be a graph and let c be a vertex coloring assigning to each vertex of G a color, i.e. a positive integer. By C we denote the set of colors of the vertices of G. Now $S \subseteq V$ is a *tropical set* of the vertex-colored graph (G, c) if all colors of G appear in S. Clearly tropicality can be combined with any property of a subset of vertices in a graph, like tropical independent sets, tropical vertex covers, etc. In this paper, we study the combination being closest to *Graph Motif*, namely tropical connected sets. The problem *Tropical Connected Set* takes as input a vertex-colored graph and the task is to find a tropical connected subset of vertices S of minimum size. It is worth mentioning that the problem generalizes in a natural way well-known NP-complete problems like *Steiner Tree* and *Connected Dominating Set*.

Angles d'Auriac et al. studied the complexity of *Tropical Connected Set*. With a reduction from the well-known NP-complete problem Dominating Set, they proved that finding a minimum tropical connected set is NP-complete, even when restricted to trees of height three [3]. Furthermore they showed that *Tropical Connected Set* also remains NP-complete on split graphs and on interval graphs [3].

Our main contributions are two exact algorithms solving the NP-hard problem *Tropical Connected Set*. The running time is $\mathcal{O}^*(1.5359^n)$ when the inputs are general vertex-colored graphs, and it is $\mathcal{O}^*(1.2721^n)$ when the inputs are restricted to vertex-colored trees. We also show that *Tropical Connected Set* on trees has no sub-exponential algorithm unless the Exponential Time Hypothesis fails; thus the existence of such a sub-exponential algorithm is considered very unlikely.

2. Preliminaries

For graph-theoretic notions not defined in the paper we refer to the monograph of Diestel [10]. Throughout this paper, we denote by $G = (V, E)$ an undirected graph, and by $T = (V, E)$ an undirected tree with vertex set V and edge set E. We adopt the convention $n = |V|$ and $m = |E|$. For a subset $X \subseteq V$ of vertices, we denote by $G[X]$ the subgraph of G induced by X. For a vertex $v \in V$ of G, we denote by $N(v)$ the set of all neighbors of v; and we let $N[v] = N(v) \cup \{v\}$. For every $X \subseteq V$, we denote by $N[X] = \bigcup_{x \in X} N[x]$ the closed neighborhood of X, and by $N(X) = N[X] \setminus X$ the open neighborhood of X. A vertex set $S \subseteq V$ of G is connected if the subgraph $G[S]$ is connected. Let $G = (V, E)$ be a graph, and let $c : V \to \mathbb{N}$ be a (not necessarily proper) coloring of G. Then we call (G, c) a vertex-colored graph and $C = \{c(v) : v \in V\}$ the set of colors of G. For a subset of vertices S of a vertex-colored graph (G, c), we denote by $c(S) = \{c(v) : v \in S\}$ the set of colors of S, and we call the set S tropical if $c(S) = C$.

We denote by $l_1(G)$ the number of colors appearing exactly once in a vertex-colored graph (G, c), and by $l_2(G)$ the number of colors appearing at least twice in (G, c). A connected component U of a disconnected graph G has a tropical connected set if and only if all colors of the graph G appear in U. Hence (G, c) has no tropical connected set if and only if none of its components contains all colors of G. Thus it suffices to find tropical connected sets in connected graphs.

Throughout the paper all graphs and trees are vertex-colored and if there is no ambiguity they will often be denoted by G and T instead of (G, c) and (T, c) respectively.

In this paper, we focus on the Tropical connected set problem:

Tropical connected set
Input: Graph $G = (V, E)$ with a coloring $c : V \to \mathbb{N}$ and set of colors C.
Question: Find a minimum size subset $S \subseteq V$ such that $G[S]$ is connected, and S contains at least one vertex of each color in C.

3. An exact exponential algorithm for general graphs

This section is devoted to the design and analysis of an exact algorithm for Tropical connected set. A naive brute force algorithm checking every vertex subset if it is tropical connected solves this problem in $\mathcal{O}^*(2^n)$ time. Using simple reductions to Connected red–blue dominating set and Steiner tree, using the balancing technique we construct an algorithm computing a minimum tropical connected set in a graph in time $\mathcal{O}^*(1.5359^n)$. Let us recall the definition of these two problems:

Steiner tree
Input: Graph $G = (V, E)$, weight function $w : E \to \mathbb{N}$, set of terminals $K \subseteq V$.
Question: Find a connected subtree $T = (V', E')$ of G, with $V' \subseteq V$ and $E' \subseteq E$, such that $K \subseteq V'$ and $\sum_{e \in E'} w(e)$ is minimum.

Connected red–blue dominating set
Input: Graph $G = (R, B, E)$ where vertices are colored either red (vertices in R) or blue (vertices in B).
Question: Find the smallest subset $S \subseteq R$ of red vertices such that $G[S]$ is connected, and every vertex in B has at least one neighbor in S, that is $B \subseteq N(S)$.

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