Theoretical Computer Science ••• (••••) •••–•••

Contents lists available at ScienceDirect

Theoretical Computer Science

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The firefighter problem: Further steps in understanding its complexity ☆

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ARTICLE INFO

Article history:
Received 27 November 2015
Received in revised form 1 February 2017
Accepted 10 March 2017
Available online xxxx
Communicated by F.V. Fomin

Keywords:
Firefighter problem
Parameterized complexity
Pathwidth
Trees

ABSTRACT

We consider the complexity of the firefighter problem where a budget of $b \geq 1$ firefighters are available at each time step. This problem is known to be NP-complete even on trees of degree at most three and $b = 1$ [14] and on trees of bounded degree $(b + 3)$ for any fixed $b \geq 2$ [4].

In this paper we provide further insight into the complexity landscape of the problem by showing a complexity dichotomy result with respect to the parameters pathwidth and maximum degree of the input graph. More precisely, first, we prove that the problem is NP-complete even on trees of pathwidth at most three for any $b \geq 1$. Then we show that the problem turns out to be fixed parameter-tractable with respect to the combined parameter "pathwidth" and "maximum degree" of the input graph. Finally, we show that the problem remains NP-complete on very dense graphs, namely co-bipartite graphs, but is fixed-parameter tractable with respect to the parameter "cluster vertex deletion".

1. Introduction

The firefighter problem was introduced by Hartnell [17] and received considerable attention in a series of papers [2,7,11,12,14,18,20,21,23,24]. In its original version, a fire breaks out at some vertex of a given graph. At each time step, one vertex can be protected by a firefighter and then the fire spreads to all unprotected neighbors of the vertices on fire. The process ends when the fire can no longer spread. At the end all vertices that are not on fire are considered as saved. The objective is at each time step to choose a vertex which is protected by a firefighter such that a maximum number of vertices in the graph is saved at the end of the process. In this paper we consider a more general version which allows us to protect $b \geq 1$ vertices at each step (the value b is called *budget*).

The original firefighter problem was proved to be NP-hard for bipartite graphs [23], cubic graphs [21] and unit disk graphs [15]. Finbow et al. [14] showed that the problem is NP-hard even on trees. More precisely, they proved the following dichotomy theorem: the problem is NP-hard even for trees of maximum degree three and it is solvable in polynomial-time for graphs with maximum degree three, provided that the fire breaks out at a vertex of degree at most two. Furthermore, the problem is polynomial-time solvable for caterpillars and so-called P-trees [23]. Later, Bazgan et al. [4] extended the previous results by showing that the general firefighter problem is NP-hard even for trees of maximum degree $(b + 3)$ for

☆ The preliminary results of this publication were presented at IPEC 2014 [9].

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[1] A major part of this work was done during a three-month visit of the University of Portsmouth supported by the ERASMUS program.

http://dx.doi.org/10.1016/j.tcs.2017.03.004
0304-3975/

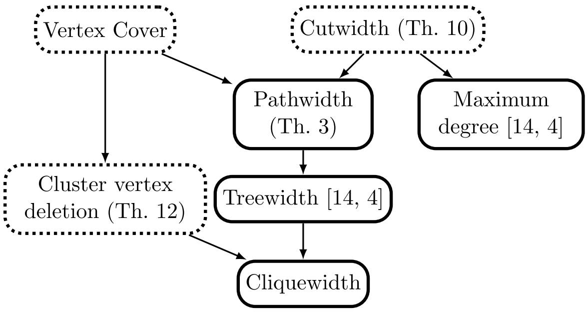

Fig. 1. The parameterized complexity of the Firefighter problem with respect to some structural graph parameters. An arc from a parameter k_2 to a parameter k_1 means that there exists some function h such that $k_1 \le h(k_2)$. For any fixed budget, a dotted rectangle means fixed-parameter tractability for this parameter and a thick rectangle means NP-hardness even for constant values of this parameter.

any fixed budget $b \ge 2$ and polynomial-time solvable on k-caterpillars. From the approximation point of view, the problem is $\frac{e}{e-1}$-approximable on trees ($\frac{e}{e-1} \approx 1.5819$) [7] and it is not $n^{1-\varepsilon}$-approximable on general graphs for any $\varepsilon > 0$ unless $P = NP$ [2]. Moreover for trees in which each non-leaf vertex has at most four neighbors, the firefighter problem is 1.3997-approximable [20]. Very recently, a significant progress has been achieved on the approximability status of the problem for trees. Chalermsook et al. claimed in [8] that the integrability gap of the standard LP relaxation can be arbitrarily close to $\frac{e}{e-1}$ and finally Adjiashvili et al. claimed to prove a PTAS for the firefighter problem on trees [1]. Costa et al. [11] extended the $\frac{e}{e-1}$-approximation algorithm on trees to the case where the fire breaks out at $f > 1$ vertices and $b > 1$ firefighters are available at each step. From a parameterized perspective, the problem is W[1]-hard with respect to the natural parameters "number of saved vertices" and "number of burned vertices" [3]. Furthermore, it admits an $O(2^\tau k\tau)$-size kernel where τ is the minimum vertex cover of the input graph and k the number of burned vertices [3]. Cai et al. [7] presented first fixed-parameter tractable algorithms and polynomial-size kernels for trees for each of the following parameters: "number of saved vertices", "number of saved leaves", "number of burned vertices", and "number of protected vertices".

In this paper we provide a complexity dichotomy result of the problem with respect to the parameters maximum degree and pathwidth of the input graph (see Fig. 1). In Section 2 we first provide the formal definition of the problem as well as some preliminaries. In Section 3 we extend the hardness results on trees by proving that the problem is also NP-complete on trees of pathwidth three. The presented proof is also a simpler proof of the NP-completeness of the problem on trees. In Section 4 we devise a parameterized algorithm with respect to the combined parameter "pathwidth" and "maximum degree" of the input graph. In Section 5 we show that the problem is also NP-hard on co-bipartite graphs which are very dense graphs, but fixed-parameter tractable with respect parameter "cluster vertex deletion" (cvd). This last result strengthens the previous $O(2^\tau k\tau)$-size kernel as it suppresses the dependence with k and the cvd number is smaller than the vertex cover number τ. The conclusion is given in Section 6.

2. Preliminaries

Graph terminology Let $G = (V, E)$ be an *undirected graph* of order n. For a subset $S \subseteq V$, $G[S]$ is the induced subgraph of G. The *neighborhood* of a vertex $v \in V$, denoted by $N(v)$, is the set of all neighbors of v. For a vertex set $V' \subseteq V$ we define $N_{V'}(v) = N(v) \cap V'$. We denote by $N^k(v)$ the set of vertices that are at distance at most k from v. The *degree* of a vertex v is denoted by $\deg_G(v)$ and the *maximum degree* of the graph G is denoted by $\Delta(G)$.

A *linear layout* of G is a bijection $\pi : V \to \{1, \dots, n\}$. For convenience, we express π by the list $L = (v_1, \dots, v_n)$ where $\pi(v_i) = i$. Given a linear layout L, we denote the distance between two vertices in L by $d_L(v_i, v_j) = j - i$.

The *cutwidth* $\mathrm{cw}(G)$ of G is the minimum $k \in \mathbb{N}$ such that the vertices of G can be arranged in a linear layout $L = (v_1, \dots, v_n)$ in such a way that, for every $i \in \{1, \dots, n-1\}$, there are at most k edges between $\{v_1, \dots, v_i\}$ and $\{v_{i+1}, \dots, v_n\}$.

The *bandwidth* $\mathrm{bw}(G)$ of G is the minimum $k \in \mathbb{N}$ such that the vertices of G can be arranged in a linear layout $L = (v_1, \dots, v_n)$ so that $|d_L(v_i, v_j)| \le k$ for every edge v_iv_j of G.

A path decomposition $\mathcal{P}$ of G is a pair $(P, \mathcal{H})$ where P is a path with node set X and $\mathcal{H} = \{H_x : x \in X\}$ is a family of subsets of V such that the following conditions are met

1. $\bigcup_{x \in X} H_x = V$.
2. For each $uv \in E$ there is an $x \in X$ with $u, v \in H_x$.
3. For each $v \in V$, the set of nodes $\{x : x \in X$ and $v \in H_x\}$ induces a subpath of P.

The width of a path decomposition $\mathcal{P}$ is $\max_{x \in X} |H_x| - 1$. The pathwidth $\mathrm{pw}(G)$ of a graph G is the minimum width over all possible path decompositions of G.

We may skip the argument of $\mathrm{pw}(G)$, $\mathrm{cw}(G)$, $\mathrm{bw}(G)$ and $\Delta(G)$ if the graph G is clear from the context.

A *star* is a tree consisting of one vertex, called the *center* of the star, adjacent to all the other vertices.