Theoretical Computer Science 670 (2017) 33–44

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Dealing with 4-variables by resolution: An improved MaxSAT algorithm ☆

Jianer Chen [a,b], Chao Xu [a], Jianxin Wang [a,*]

[a] School of Information Science and Engineering, Central South University, Changsha, 410083, China
[b] Department of Computer Science and Engineering, Texas A&M University, College Station, TX 77843-3112, USA

ARTICLE INFO

Article history:
Received 23 July 2016
Received in revised form 6 January 2017
Accepted 24 January 2017
Available online 31 January 2017
Communicated by F.V. Fomin

Keywords:
Maximum satisfiability
Parameterized algorithm
Branch and bound
The resolution principle

ABSTRACT

We study techniques for solving the MAXIMUM SATISFIABILITY problem (MAXSAT). Our focus is on variables of degree 4. We identify cases for degree-4 variables and show how the resolution principle and the kernelization techniques can be nicely integrated to achieve more efficient algorithms for the MAXSAT problem. As a result, we present an algorithm of time $O^*(1.3248^k)$ for the MAXSAT problem, improving the previous best upper bound $O^*(1.358^k)$ by Ivan Bliznets and Alexander Golovnev.

1. Introduction

The SATISFIABILITY problem (SAT) is of fundamental importance in computer science and information technology [4]. Its optimization version, the MAXIMUM SATISFIABILITY problem (MAXSAT), plays a similar role in the study of computational optimization, in particular in the study of approximation algorithms [11]. Since the problems are NP-hard [9], different algorithmic approaches, including heuristic algorithms (e.g., [10,15]), approximation algorithms (e.g., [2,21]), and exact and parameterized algorithms (e.g., [5,6,17]), have been extensively studied.

The main result of the current paper is an improved parameterized algorithm for the MAXSAT problem. Formally, the (parameterized) MAXSAT problem is defined as follows.[1]

> MAXSAT: Given a CNF formula F and an integer k (the *parameter*), is there an assignment to the variables in F that satisfies at least k clauses in F?

It is known that the MAXSAT problem is fixed-parameter tractable, i.e., it is solvable in time $O^*(f(k))$ for a function f that only depends on the parameter k.[2] The research on parameterized algorithms for the MAXSAT problem has been focused

☆ A preliminary version of this article appeared in *Proceedings of the 14th International Symposium on Algorithms and Data Structures* (WADS'15). This work is supported by the National Natural Science Foundation of China, under Grant Nos. 61420106009, 61672536, 61472449 and 71221061.

* Corresponding author.
E-mail addresses: jianer@cse.tamu.edu.cn (J. Chen), xuchaofay@csu.edu.cn (C. Xu), jxwang@csu.edu.cn (J. Wang).

[1] We remark that there is a variation of the MAXSAT problem, that asks whether there is an assignment to satisfy at least $k + m/2$ clauses in a CNF formula with m clauses [14]. This variation has also drawn significant attention.

[2] Following the current convention in the research in exact and parameterized algorithms, we will use the notation $O^*(f(k))$ to denote the bound $f(k)n^{O(1)}$, where n is the instance size.

http://dx.doi.org/10.1016/j.tcs.2017.01.020
0304-3975/

Table 1
Progress in MaxSAT algorithms.

Bound	Reference	Year
$O^*(1.618^k)$	Mahajan, Raman [14]	1999
$O^*(1.400^k)$	Niedermeier, Rossmanith [16]	2000
$O^*(1.381^k)$	Bansal, Raman [3]	1999
$O^*(1.370^k)$	Chen, Kanj [6]	2004
$O^*(1.358^k)$	Bliznets, Golovnev [5]	2012
$O^*(1.3248^k)$	this paper	2016

on improving the upper bound on the function f, with an impressive list of improvements. Table 1 summarizes the progress in the research. For comparison, we have also included our result in the current paper in the table.

Most algorithms for SAT and MaxSAT are based on the branch-and-bound process [10]. The *Strong Exponential Time Hypothesis* conjectures that the SAT problem cannot be solved in time $O^*(2^{cn})$ for any constant $c < 1$, where n is the number of variables in the input CNF formula [12]. The hypothesis indicates, to some extent, a popular opinion that branch-and-bound is perhaps unavoidable in solving the SAT problem and its variations.

Therefore, it has become critical how to branch more effectively in algorithms solving the SAT and MaxSAT problems. In particular, all existing parameterized algorithms for MaxSAT and most known algorithms for SAT have been focused on more effective branching strategies to further improve the algorithm complexity. Define the *degree* of a variable x in a CNF formula F to be the number of times x and $\bar{x}$ appear in the formula. For MaxSAT, it is well-known that branching on variables of large degree will be sufficiently effective. On the other hand, variables of degree bounded by 2 can be handled efficiently based on the resolution principle [8]. In 2012, Bliznets and Golovnev [5] proposed new strategies for branching on degree-3 variables more effectively and improved Chen and Kanj's algorithm [6], which had stood as the best MaxSAT algorithm for eight years.

To further improve the algorithm complexity for MaxSAT, the next bottleneck is on degree-4 variables. Degree-4 variables seem neither to have a large enough degree to support direct branchings of sufficient efficiency, nor to have simple enough structures that allow feasible case-by-case analysis to yield more efficient manipulations. In fact, degree-4 variables are the sources for the worst branching cases in Chen–Kanj's algorithm (case 3.10 in [6]) as well as in Bliznets–Golovnev's algorithm (Theorem 5, step 10 in [5]).

A contribution of the current paper is to show how the resolution principle [8] can be used in handling degree-4 variables in solving the MaxSAT problem. It has been well-known that the resolution principle is a very powerful tool in solving the SAT problem [8]. In particular, variable resolutions in a CNF formula preserve the satisfiability of the formula. Unfortunately, variable resolutions cannot be used directly in solving the MaxSAT problem in general case because they do not provide a predictable decreasing in the maximum number of clauses in the CNF formulas that can be satisfied by an assignment. In particular, for a degree-4 variable x in a CNF formula F for which an optimal assignment satisfies k clauses, the resolution on x may result in CNF formulas for which optimal assignments satisfy $k-3$, $k-2$, $k-1$, and k clauses, respectively.

We identify cases for degree-4 variables and show how the resolution principle can be applied efficiently on these cases (see our reduction rules R-Rules 6–7). This technique helps us to eliminate the structures that do not support efficient branchings. We also show how the resolution principle and kernelization algorithms of parameterized problems are nicely integrated. Note that resolutions may significantly increase the size and the number of clauses in a formula. However, it turns out to be not a concern for algorithms for MaxSAT: MaxSAT has a polynomial-time kernelization algorithm [6] that can bound the size of the formula F by $O(k^2)$ in an instance (F, k) of MaxSAT. Therefore, the resolution principle can be used whenever it is applicable – once the formula size gets too large, we can simply use the kernelization algorithm to reduce the formula size. In fact, one of our reduction rules (R-Rule 7) does not even decrease the parameter value, which, however, can only be applied polynomial many times because of the kernelization of MaxSAT.

A nice approach suggested by Bliznets and Golovnev [5] is to transform solving MaxSAT on a class of special instances into solving the Set-Cover problem. However, the method proposed in [5] is not efficient enough to achieve our bound. For this, we introduce new branching rules that are sufficiently efficient and further reduce the instances to an even more restricted form. In particular, we show how to eliminate all clauses of size 2 and 3. The restricted form of the instances allows us to apply more powerful techniques in randomized algorithms and in derandomization [21] to derive tighter lower bounds on the instances of MaxSAT, which makes it become possible to use more effectively the existing algorithm for Set-Cover [18].

The paper is organized as follows. Section 2 provides preliminaries and necessary definitions. Section 3 describes our reduction rules, which are the polynomial-time processes that can be used to simplify the problem instances. Branching rules are given in Section 4 that are applied on instances of specified structures. A complete algorithm is presented in Section 5. Conclusions and remarks are given in Section 6 where we also discuss the difficulties of further improving the results presented in the current paper.