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The complexity of counting quantifiers on equality languages [☆]

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ABSTRACT

An equality language is a relational structure with infinite domain whose relations are first-order definable in equality. We classify the extensions of the quantified constraint satisfaction problem over equality languages in which the native existential and universal quantifiers are augmented by some subset of counting quantifiers. In doing this, we find ourselves in various worlds in which dichotomies or trichotomies subsist.

1. Introduction

The *constraint satisfaction problem* CSP, much studied in artificial intelligence, is known to admit several equivalent formulations, two of the best known of which are the query evaluation of primitive positive (pp) sentences – those involving only existential quantification and conjunction – and the homomorphism problem (see, e.g., [1]). The CSP is NP-complete in general, and a great deal of effort has been expended in classifying the complexity of CSP(Γ) across fixed, finite *constraint languages* Γ. Notably it is conjectured [2,3] that for all such finite Γ, the problem CSP(Γ) is in P or NP-complete. While this has not been settled in general, a number of partial results are known – e.g. over structures of size at most three [4,5] and over smooth digraphs [6,7].

A popular generalisation of the CSP involves considering the query evaluation problem for *positive Horn* logic – involving only the two quantifiers, $\exists$ and $\forall$, together with conjunction. The resulting *quantified constraint satisfaction problems* QCSP(Γ) allow for a broader class, used in artificial intelligence to capture non-monotonic reasoning [8], whose complexities rise to Pspace-completeness.

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Theorem	Subsets
1	$\emptyset$
2	$\{\forall^{>2}\}$, $\{\forall^{>1}, \forall^{>2}\}$
3	$\{\forall^{>1}\}$
4	$\{\exists^{\geq 2}\}$
5	$\{\forall^{>2}, \exists^{\geq 2}\}$, $\{\forall^{>1}, \forall^{>2}, \exists^{\geq 2}\}$
6	$\{\forall^{>1}, \exists^{\geq 2}\}$

Fig. 1. Classification theorems linked to canonical subsets of $\{\exists^{\geq 2}, \forall^{>1}, \forall^{>2}\}$.

Once upon a time, Bodirsky and Kára gave a systematic classification for CSP(Γ), where Γ consists of relations first-order (fo) definable in equality, over some countably infinite domain [9]. These so-called *equality languages* Γ display dichotomy between those for which CSP(Γ) is in P and those for which it is NP-complete. As explained in [10], equality languages form a base case in the pursuit of grander classifications across first-order definitions over more complicated structures. Pursuing this line of investigation, Bodirsky and Chen gave a trichotomy for QCSP(Γ), where Γ is an equality language – each problem being either in P, NP-complete or co-NP-hard [11]. In the conference version of that paper, the trichotomy was claimed to be across P, NP-complete or Pspace-complete [12], but the proof in the tricky case of $x = y \rightarrow y = z$ was flawed, and so in the journal version this became the weaker co-NP-hard (and in Pspace). The trichotomy is thus imperfect, as most of the co-NP-hard cases are known to be Pspace-complete. Indeed, $x = y \rightarrow y = z$ would be the only open case, if it were Pspace-complete [13].

Working hypothesis. QCSP($x = y \rightarrow y = z$) is Pspace-complete.

Thus the assumption of the working hypothesis would restore the trichotomy to the P, NP-complete or Pspace-complete as stated in [12].

In this paper, we consider the generalisation of the QCSP with counting quantifiers, as pioneered in the recent paper [14]. In [14], the domains of Γ were of finite size n, so the extant quantifiers $\exists^{\geq 1} = \exists$ and $\exists^{\geq n} = \forall$ were augmented with quantifiers of the form $\exists^{\geq j}$, which allow one to assert the existence of at least j elements such that the ensuing property holds. In the world of infinite domains, it makes sense to permit not only quantification above the finite with $\exists^{\geq j}$, but also quantification *below the co-finite* with $\forall^{>j}$, whose intended meaning is that the property holds for all but (at most) j elements of the domain. Thus, $\forall = \forall^{>0}$. Counting quantifiers of the form $\exists^{\geq j}$ have been extensively studied in finite model theory (see [15,16]), where the focus is on supplementing the descriptive power of various logics. Quantifiers of the form $\forall^{>j}$ appear rare in computer science but these quantifiers together with $\exists^{\geq j}$ are termed *hemilogical* when they appear in [17]. Of broader interest is the *majority quantifier* $\exists^{\geq n/2}$ (on a structure of domain size n), which sits broadly midway between $\exists$ and $\forall$. Majority quantifiers are studied across diverse fields of logic and have various practical applications, e.g. in cognitive appraisal and voting theory [18,19]. They have also been studied in computational complexity since at least [20] (see also [15]).

We study extensions of QCSP(Γ) in which the input sentence to be evaluated on Γ remains positive conjunctive in its quantifier-free part, but is quantified by various counting quantifiers. For $X \subseteq \{\exists^{\geq 1}, \exists^{\geq 2}, \ldots, \forall^{>0}, \forall^{>1}, \ldots\}$, $X \supseteq \{\exists^{\geq 1}, \forall^{>0}\}$, the X-CSP(Γ) takes as input a sentence given by a conjunction of atoms quantified by quantifiers appearing in X. It then asks whether this sentence is true on Γ.

Equality languages admit quantifier elimination of $\forall$ and $\exists$, that is any relation first-order definable in equality is already quantifier-free definable, say as a CNF. An equality language Γ is

- *trivial* if all its relations may be given as a conjunction of equalities,
- *specially negative* if the class of relations over Γ, closed under definability in the positive conjunctive logic with quantifiers among $\{\exists, \forall, \forall^{>1}\}$, does not contain the formula $x \neq y \vee y \neq z$,
- *negative* if all its relations may be given as a conjunction of equalities and disjunctions of disequalities, and
- *positive* if all its relations may be given as a conjunction of disjunctions of equalities.

Similarly, we might use these adjectives on the relations within the equality language. We observe the containments of trivial languages within specially negative languages within negative languages. Further, it is proved in [11] (Proposition 7.3) that the positive languages that are not trivial are precisely the positive languages that are not negative.

Our main results are a complete panoply of classifications for $X \supseteq \{\exists^{\geq 1}, \forall^{>0}\}$, that is the augmentation of $\exists$ and $\forall$ with the more exotic counting quantifiers. It will be seen that the quantifiers $\exists^{\geq 2}, \exists^{\geq 3}, \ldots$ more or less behave as one another and similarly with $\forall^{>2}, \forall^{>3}, \ldots$. However, $\forall^{>1}$ is special and thus our task of classifications for X amounts to choosing subsets of $\{\exists^{\geq 2}, \forall^{>1}, \forall^{>2}\}$ with which to augment $\{\exists^{\geq 1}, \forall^{>0}\}$. A priori there are then eight possibilities, but twice we will see $\forall^{>1}$ being "subsumed" by $\forall^{>2}$. Thus we will give *six distinct classification theorems*: three dichotomies and three trichotomies (one of which is that of [11]). In Fig. 1, these classification theorems are linked to their canonical subsets of $\{\exists^{\geq 2}, \forall^{>1}, \forall^{>2}\}$.