Theoretical Computer Science ••• (••••) •••–•••

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Hardness results for stable exchange problems

Zsuzsa Mészáros-Karkus

Department of Operations Research, Eötvös Loránd University, Pázmány Péter sétány 1/C, H-1117 Budapest, Hungary

A R T I C L E I N F O

Article history:
Received 6 October 2016
Received in revised form 23 December 2016
Accepted 30 January 2017
Available online xxxx
Communicated by P.G. Spirakis

Keywords:
Stable exchange
Kidney exchange
Computational complexity
Stable partition

A B S T R A C T

In this paper, we study variants of the stable exchange problem which can be viewed as a model for kidney exchange. The b-way stable l-way exchange problem is a generalization of the stable roommates problem. For $b = l = 3$, Biró and McDermid (2010) [1] proved that the problem is NP-complete and asked whether a polynomial time algorithm exists for $b = 2$, $l = 3$. We prove that the problem is NP-complete and it is W[1]-hard with the number of 3-cycles in the exchange as a parameter. We answer a question of Biró (2007) [2] by proving that it is NP-hard to maximize the number of covered nodes in a stable exchange. We also prove some related results on strong stability and approximation.

1. Introduction

Given a simple digraph $D = (V, A)$, a set of node-disjoint directed cycles is called an **exchange**. In the stable exchange problem, the input is a preference system, which consist of a simple digraph $D = (V, A)$ and a strictly ordered preference list L_v of the out-neighbors of v for every $v \in V$. We say that u gets v in the exchange E if uv is an arc of one of the directed cycles in E. We say that $v \in V$ is **covered** by the exchange E if v belongs to a cycle in E.

An exchange E is called

- **stable** if for every directed cycle C, there exists an arc $uv \in C$ such that uv is in E or u prefers its out-neighbor in E over v.
- **strongly stable** if for every directed cycle C not in E, there exists an arc $uv \in C$ such that u prefers its out-neighbor in E over v.

In both cases, the node set of a directed cycle violating the described conditions is called a **blocking coalition**.

An important motivation of this model is kidney exchange. (This was first described in [17].) Currently the best known treatment for kidney failure is transplantation. Since there are a large number of people on the deceased donor waiting list, the more efficient solution is living donation. However, a kidney of a willing living donor is often not suitable for the patient for immunological reasons. Therefore incompatible patient-donor pairs might want to exchange kidneys with other pairs in the same situation. Kidney exchanges have been organized in several countries; for an overview of the different approaches, see [4,17,20]. In the model described above, the nodes of the digraph correspond to the incompatible patient-donor pairs and $uv \in A$ if and only if the kidney of the donor corresponding to v is suitable for the patient corresponding to u. Each patient has a strict preference order over the kidneys suitable for him. In an exchange, the patient-donor pairs exchange kidneys backwards along the cycles.

E-mail address: karkuszsuzsi@gmail.com.

http://dx.doi.org/10.1016/j.tcs.2017.01.023
0304-3975/

Shapley and Scarf [19] showed that the stable exchange problem (SE) is always solvable, and a stable exchange can be found by the Top Trading Cycles (TTC) algorithm proposed by Gale.

In case of kidney exchanges, the cycles in the exchange should be short, since all operations along a cycle have to be carried out at the same time (to avoid someone backing out).

Definition 1. If all the cycles in the exchange have length at most l, we call it an **l-way exchange**. An exchange is called **b-way stable** if there is no blocking coalition of size at most b. The definition is analogous for strong stability.

Biró and McDermid [1] proved that the problem of deciding whether a 3-way stable 3-way exchange exists is NP-complete, and asked whether a polynomial time algorithm exists for the problem of deciding whether a 2-way stable 3-way exchange exists. In section 2 we prove that the problem is NP-complete and it is W[1]-hard with the number of 3-cycles in the exchange as a parameter, even in complete digraphs. We also prove that the problem of deciding whether a b-way strongly stable l-way exchange exists is NP-complete for any $b \geq 2,\ l \geq 3$ and the same holds for b-way stable l-way exchanges and stable l-way exchanges.

An instance might admit more than one stable exchange; therefore, it is a natural goal to maximize the number of covered nodes in the exchange. The complexity of this problem was mentioned as an open problem in [2] as well as the same question for 2-way stable exchanges.

Definition 2. An exchange is called **complete** if it covers every node.

In section 3, we show that deciding if an instance admits a complete stable exchange is NP-complete and the same holds for b-way stable exchanges for any $b \geq 2$. Roth and Postlewaite [18] proved that the exchange found by the TTC algorithm is strongly stable and it is the only strongly stable solution. However, there might be more than one b-way strongly stable exchange. We prove that deciding if an instance admits a complete b-way strongly stable exchange is NP-complete for any $b \geq 2$.

Definition 3. A digraph $D = (V, A)$ is called **symmetric**, if for every $uv \in A$, $vu \in A$ as well. We call two arcs in opposite directions between the same two nodes a **bidirected edge**. A simple digraph is **complete**, if there is a bidirected edge between any two of its nodes.

We show that if the digraph is symmetric, then TTC is a $\frac{1}{2}$-approximation algorithm, while the stable partition algorithm is a $\frac{2}{3}$-approximation algorithm for maximizing the number of covered nodes in a 2-way (strongly) stable exchange. All the NP-hardness reductions are from the k-CLIQUE IN k-PARTITE GRAPH problem, which is specified as follows:

Instance: An integer k, and a k-partite graph $G = (V_1 \cup V_2 \cup \ldots \cup V_k, E)$, such that $\lvert V_i \rvert$ is odd and $\lvert V_i \rvert \geq 5$ for $i = 1, \ldots, k$. Question: Is there a clique of size k in G?

The NP-completeness of this problem was implicitly proved in [11], and the W[1]-completeness of the problem was proved in [9].

1.1. Related work

An instance of the stable marriage problem (SM) consists of n men and n women. Each person has a strictly ordered preference list containing all members of the opposite sex. The problem is to find a matching which is stable in the sense that there is no blocking pair, i.e. a man and a woman who prefer each other over their partners in the matching. The Gale-Shapley algorithm [10] always finds a stable matching in an instance of SM.

In the stable roommates problem (SR) there are $2n$ persons, each of whom ranks all the others in strict order of preference. The goal is to find a complete stable matching. Gale and Shapley [10] gave an instance of SR for which no stable matching is possible. Irving [12] proposed an $O(n^2)$ time algorithm which finds a complete stable matching if there is one, or reports that none exists.

The stable roommates with incomplete lists problem (SRI) is a generalization of SR, where each person's preference list only contains his acceptable partners. The problem can be represented by a graph, where there is an edge between two persons if and only if they are acceptable to each other. Here the number of people is not necessarily even, and the stable matching does not need to be complete. However, the same persons are matched in every stable matching and Irving's algorithm can be extended to SRI [13].

The stable exchange problem and the definition of b-way stable l-way exchanges have already been described above. We may assume that if $uv \in A$ in an instance of the 2-way stable 2-way exchange problem, then $vu \in A$, since otherwise uv does not belong to any 2-cycle or blocking coalition. Therefore, the 2-way stable 2-way exchange problem is equivalent to

Please cite this article in press as: Zs. Mészáros-Karkus, Hardness results for stable exchange problems, Theoret. Comput. Sci. (2017), http://dx.doi.org/10.1016/j.tcs.2017.01.023