Theoretical Computer Science 662 (2017) 40–47

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

Forbidden directed minors and Kelly-width

Shiva Kintali [a,*], Qiuyi Zhang [b]

[a] Department of Computer Science, Princeton University, Princeton, NJ 08540, United States

[b] Mathematics Department, Princeton University, Princeton, NJ 08540, United States

ARTICLE INFO

Article history:
Received 2 April 2016
Received in revised form 28 October 2016
Accepted 6 December 2016
Available online 2 January 2017
Communicated by A. Marchetti-Spaccamela

Keywords:
Forbidden minors
Graph minors
Kelly-width
Partial k-DAGs
Treewidth

ABSTRACT

Partial k-DAGs are generalizations of partial k-trees. Partial k-trees are undirected graphs with bounded treewidth, whereas partial k-DAGs are digraphs with bounded Kelly-width. It is well-known that an undirected graph is a partial 1-tree if and only if it has no K_3 minor. In this paper, we prove that partial 1-DAGs are characterized by three forbidden directed minors, K_3, N_4 and M_5.

1. Introduction

Treewidth and pathwidth (and their associated decompositions) played a crucial role in the development of graph minor theory [19–22] and proved to be algorithmically and structurally important graph parameters. Treewidth (resp. pathwidth) measures how similar a graph is to a tree (resp. path). Treewidth has several equivalent characterizations in terms of elimination orderings, elimination trees (used in symmetric matrix factorization), partial k-trees and cops and (visible and eager) robber games. Several problems that are NP-hard on general graphs are solvable in polynomial time (some even in linear time, some are fixed-parameter tractable) on graphs of bounded treewidth by using dynamic programming on a tree-decomposition of the input graph. Similarly pathwidth has several equivalent characterizations in terms of vertex separation number, node searching number, interval thickness (i.e., one less than the maximum clique size in an interval supergraph) and cops and (invisible and eager) robber games. Several problems that are NP-hard on general graphs are efficiently solvable on graphs of bounded pathwidth.

Motivated by the success of treewidth and pathwidth, efforts have been made to generalize these concepts to digraphs. Directed treewidth [13], D-width [23], DAG-width [5,18,4] and Kelly-width [11] are some such notions which generalize treewidth, whereas directed pathwidth [3] generalizes pathwidth. All these parameters have associated decompositions called arboreal decompositions, D-decompositions, DAG-decompositions, Kelly-decompositions and directed path decompositions respectively. Hamitonian cycle, Hamiltonian path, k-disjoint paths and weighted disjoint paths are solvable in polynomial time on digraphs of bounded directed treewidth. In addition to these problems, parity games are solvable in polynomial time on digraphs of bounded DAG-width. Directed treewidth and DAG-width started an interesting line of research but they suffer from some disadvantages. Directed treewidth is not monotone under butterfly minors (a natural set

* Corresponding author.
E-mail addresses: kintali@cs.princeton.edu (S. Kintali), qiuyiz@princeton.edu (Q. Zhang).

http://dx.doi.org/10.1016/j.tcs.2016.12.008

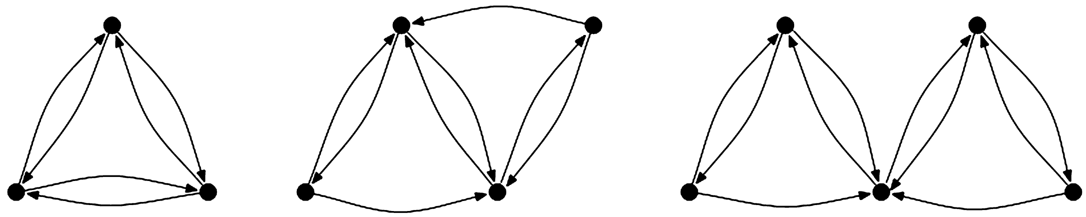

Fig. 1. The forbidden directed minors of partial 1-DAGs: K_3, N_4 and M_5.

of directed minor operations) and they do not have an *exact* characterization in terms of cops and robber games. DAG-width has an exact characterization in terms of *monotone* cops and (visible and eager) robber games. Unfortunately, (i) there is an infinite class of graphs such that every DAG-decomposition of optimal width has size super-polynomial in the size of the graph, (ii) there is no polynomial size DAG-decomposition which would approximate an optimal decomposition up to an additive constant and (iii) deciding whether the DAG-width of a given graph is at most a given constant is PSPACE-complete [1].

Hunter and Kreutzer [11] showed that Kelly-width not only generalizes treewidth but is also characterized by several equivalent notions such as directed elimination orderings, elimination DAGs [10] (used in asymmetric matrix factorization), partial k-DAGs and *monotone* cops and (invisible and inert) robber games. Hamiltonian cycle, Hamiltonian path, k-disjoint paths, weighted disjoint paths and parity games are solvable in polynomial time on digraphs of bounded Kelly-width. The size of Kelly-decompositions can be made linear and their structure is suitable for dynamic-programming-type algorithms (see [11] for more details). Hence, Kelly-width is the best known generalization of treewidth both from an algorithmic perspective and structural perspective.

The graph minor theorem [21] (i.e., undirected graphs are well-quasi-ordered under the minor relation) implies that every minor-closed family of undirected graphs has a finite set of minimal forbidden minors. In particular, it implies that for all $k \geq 0$, graphs of treewidth (or pathwidth) $\leq k$ are characterized by a finite set of forbidden minors. The graph minor theorem does not provide an algorithm to compute the set of forbidden minors for a given class of graphs. In fact, this problem is undecidable in general. The complete sets of forbidden minors are known for graph with small treewidth and pathwidth. A graph has treewidth at most one (resp. two) if and only if it is K_3-free (resp. K_4-free). Graphs of treewidth at most three are characterized by four forbidden minors (K_5, the graph of the octahedron, the graph of the pentagonal prism, and the Wagner graph) [2,24]. Graphs of pathwidth at most one (resp. two) are characterized by two (resp. 110) forbidden minors [6,14].

A natural question is "*are partial k-DAGs (i.e., digraphs of bounded Kelly-width) characterized by a finite set of forbidden directed minors*?". There is no generalization of the graph minor theorem for digraphs yet. Existing notions of directed minors (e.g. directed topological minors [12], butterfly minors [13], strong contractions [15], directed immersions [7]) do not imply well-quasi-ordering of all digraphs.

1.1. Motivation

Our results are motivated by the question "*what are the complete sets of directed forbidden minors for digraphs with small values of Kelly-width*?". We exhibit the sets of *forbidden directed minors* for digraphs with Kelly-width one and two (i.e., partial 0-DAGs and partial 1-DAGs). We prove that partial 0-DAGs are characterized by one forbidden directed minor (see Lemma 14) and partial 1-DAGs are characterized by three forbidden directed minors (see Fig. 1).

Meister et al. [17] presented a polynomial-time algorithm to recognize partial 1-DAGs (i.e., digraphs of Kelly-width two). While their result is algorithmic, our result is structural. Our result explains the hidden structures in a digraph that force higher Kelly-width.

In the undirected world, whether a fixed graph H is a minor (or a subdivision) of an input graph G can be decided in polynomial time [20]. This implies that any minor-closed class of graphs can be decided in polynomial time, given a list of the corresponding forbidden minors. Can we generalize these results to digraphs? Unfortunately the answer is NO, unless $P = NP$. There are some fixed graphs (say H) for which deciding $H \precsim G$ is NP-complete [9]. Nevertheless, it is worthwhile to ask "*for which fixed graphs H, is the* directed minor detection problem *solvable in polynomial-time?*". To study this question we first need to explore interesting classes of such graphs. Our result presents three such graphs (K_3, N_4 and M_5) that arose naturally as forbidden directed minors of partial 1-DAGs. Thus, these graphs are important in deriving not only the structural insights, but also the algorithmic insights of directed width parameters, directed minors and digraphs in general.

An interesting open problem is to prove a forbidden directed minor characterization of partial 2-DAGs (i.e., digraphs of Kelly-width three). Such a characterization seems to require substantially new ideas. Graphs of pathwidth at most one are characterized by two forbidden minors [6]. In a sequel to this paper, we proved that the digraphs with directed pathwidth at most one are characterized by a finite number of forbidden directed minors [16].