Doctopic: Algorithms, automata, complexity and games ARTICLE IN PRESS TCS:11041

Theoretical Computer Science ••• (••••) •••–•••

Contents lists available at ScienceDirect

Theoretical Computer Science

www.elsevier.com/locate/tcs

On the conjugacy class of the Fibonacci dynamical system

F. Michel Dekking [a,*], Michael S. Keane [a,b]

[a] DIAM, Delft University of Technology, Faculty EEMCS, P.O. Box 5031, 2600 GA Delft, The Netherlands
[b] New York University Shanghai, China

ARTICLE INFO

Article history:
Received 17 August 2016
Received in revised form 14 December 2016
Accepted 12 January 2017
Available online xxxx
Communicated by J. Karhumäki

Keywords:
Fibonacci word
Automatic sequences
Topological conjugacy
Symbolic dynamical system

ABSTRACT

We characterize the symbolical dynamical systems which are topologically isomorphic to the Fibonacci dynamical system. We prove that there are infinitely many injective primitive substitutions generating a dynamical system in the Fibonacci conjugacy class. In this class there are infinitely many dynamical systems not generated by a substitution. An example is the system generated by doubling the 0's in the infinite Fibonacci word.

1. Introduction

We study the Fibonacci substitution φ given by

$$\varphi: \quad 0 \to 01,\ 1 \to 0.$$

The infinite Fibonacci word w_{F} is the unique one-sided sequence (to the right) which is a fixed point of φ:

$$w_{\mathrm{F}} = 0100101001\dots.$$

We also consider one of the two two-sided fixed points x_{F} of φ^2:

$$x_{\mathrm{F}} = \dots 01001001 \cdot 0100101001 \dots.$$

The dynamical system generated by taking the orbit closure of x_{F} under the shift map σ is denoted by (X_φ, σ).

The question we will be concerned with is: what are the substitutions η which generate a symbolic dynamical system topologically isomorphic to the Fibonacci dynamical system? Here *topologically isomorphic* means that there exists a homeomorphism $\psi : X_\varphi \to X_\eta$, such that $\psi\sigma = \sigma\psi$, where we denote the shift on X_η also by σ. In this case (X_η, σ) is said to be *conjugate* to (X_φ, σ).

This question has been completely answered for the case of constant length substitutions in the paper [2]. It is remarkable that there are only finitely many injective primitive substitutions of length L which generate a system conjugate to a given substitution of length L. Here a substitution α is called *injective* if $\alpha(a) \neq \alpha(b)$ for all letters a and b from the alphabet with $a \neq b$. When we extend to the class of all substitutions, replacing L by the Perron–Frobenius eigenvalue of the

* Corresponding author.
E-mail address: F.M.Dekking@TUDelft.nl (F.M. Dekking).

http://dx.doi.org/10.1016/j.tcs.2017.01.009

Please cite this article in press as: F.M. Dekking, M.S. Keane, On the conjugacy class of the Fibonacci dynamical system, Theoret. Comput. Sci. (2017), http://dx.doi.org/10.1016/j.tcs.2017.01.009

incidence matrix of the substitution, then the conjugacy class can be infinite in general. See [5] for the case of the Thue–Morse substitution. In the present paper we will prove that there are infinitely many injective primitive substitutions with Perron–Frobenius eigenvalue $\Phi = (1+\sqrt{5})/2$ which generate a system conjugate to the Fibonacci system—see Theorem 5.1.

In the non-constant length case some new phenomena appear. If one has an injective substitution α of constant length L, then all its powers α^n will also be injective. This is no longer true in the general case. For example, consider the injective substitution ζ on the alphabet $\{1,2,3,4,5\}$ given by

$$\zeta: \quad 1 \to 12,\ 2 \to 3,\ 3 \to 45,\ 4 \to 1,\ 5 \to 23.$$

An application of Theorem 2.1 followed by a partition reshaping (see Section 4) shows that the system (X_ζ, σ) is conjugate to the Fibonacci system. However, the square of ζ is given by

$$\zeta^2: \quad 1 \to 123,\ 2 \to 45,\ 3 \to 123,\ 4 \to 12,\ 5 \to 345,$$

which is *not* injective. To deal with this undesirable phenomenon we introduce the following notion. A substitution α is called a *full rank* substitution if its incidence matrix has full rank (non-zero determinant). This is a strengthening of injectivity, because obviously a substitution which is not injective can not have full rank. Moreover, if the substitution α has full rank, then all its powers α^n will also have full rank, and thus will be injective.

Another phenomenon, which does not exist in the constant length case, is that non-primitive substitutions ζ may generate uniquely defined minimal systems conjugate to a given system. For example, consider the injective substitution ζ on the alphabet $\{1,2,3,4\}$ given by

$$\zeta: \quad 1 \to 12, \quad 2 \to 31, \quad 3 \to 4, \quad 4 \to 3.$$

With the partition reshaping technique from Section 4 one can show that the system (X_ζ, σ) is conjugate to the Fibonacci system (ignoring the system on two points generated by ζ). In the remainder of this paper we concentrate on primitive substitutions.

The structure of the paper is as follows. In Section 2 we show that all systems in the conjugacy class of the Fibonacci substitution can be obtained by letter-to-letter projections of the systems generated by so-called N-block substitutions. In Section 3 we give a very general characterization of symbolical dynamical systems in the Fibonacci conjugacy class, in the spirit of a similar result on the Toeplitz dynamical system in [4]. In Section 4 we introduce a tool which admits to turn non-injective substitutions into injective substitutions. This is used in Section 5 to show that the Fibonacci class has infinitely many primitive injective substitutions as members. In Section 6 we quickly analyze the case of a 2-symbol alphabet. Sections 7 and 8 give properties of equicontinuous factors and incidence matrices, which are used to analyze the 3-symbol case in Section 9. In the final Section 10 we show that the system obtained by doubling the 0's in the infinite Fibonacci word is conjugate to the Fibonacci dynamical system, but can not be generated by a substitution.

2. N-block systems and N-block substitutions

For any N the N-block substitution $\hat{\theta}_N$ of a substitution θ is defined on an alphabet of $p_\theta(N)$ symbols, where $p_\theta(\cdot)$ is the complexity function of the language $\mathcal{L}_\theta$ of θ (cf. [11, p. 95]). What is *not* in [11], is that this N-block substitution generates the N-block presentation of the system (X_θ, σ).

We denote the letters of the alphabet of the N-block presentation by $[a_1a_2\dots a_N]$, where $a_1a_2\dots a_N$ is an element from $\mathcal{L}_\theta^N$, the set of words of length N in the language of θ. The N-block presentation $(X_\theta^{[N]}, \sigma)$ emerges by applying an sliding block code Ψ to the sequences of X_θ, so Ψ is the map

$$\Psi(a_1a_2\dots a_N) = [a_1a_2\dots a_N].$$

We denote by ψ the induced map from X_θ to $X_\theta^{[N]}$:

$$\psi(x) = \dots \Psi(x_{-N},\dots,x_{-1})\Psi(x_{-N+1},\dots,x_0)\dots.$$

It is easy to see that ψ is a conjugacy, where the inverse is π_0 induced by the 1-block map (also denoted π_0) given by $\pi_0([a_1a_2\dots a_N]) = a_1$.

The *N-block substitution* $\hat{\theta}_N$ is defined by requiring that for each word $a_1a_2\dots a_N$ the length of $\hat{\theta}_N([a_1a_2\dots a_N])$ is equal to the length L_1 of $\theta(a_1)$, and the letters of $\hat{\theta}_N([a_1a_2\dots a_N])$ are the Ψ-codings of the first L_1 consecutive N-blocks in $\theta(a_1a_2\dots a_N)$.

Theorem 2.1. *Let $\hat{\theta}_N$ be the N-block substitution of a primitive substitution θ. Let $(X_\theta^{[N]}, \sigma)$ be the N-block presentation of the system (X_θ, σ). Then $X_\theta^{[N]} = X_{\hat{\theta}_N}$.*

Proof. Let x be a fixed point of θ, and let $y = \psi(x)$, where ψ is the N-block conjugacy, with inverse π_0. The key equation is $\pi_0\hat{\theta}_N = \theta\pi_0$. This implies

Please cite this article in press as: F.M. Dekking, M.S. Keane, On the conjugacy class of the Fibonacci dynamical system, Theoret. Comput. Sci. (2017), http://dx.doi.org/10.1016/j.tcs.2017.01.009