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# On the largest Cartesian closed category of stable domains ☆

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ABSTRACT

Let $D$ be a Scott-domain and $[D \to_c D]$ (resp., $[D \to_s D]$) its conditionally multiplicative (CM for short) (resp., stable) function space. Zhang (1996) [19] mentioned that if $[D \to_c D]$ is bounded complete, $D$ should be distributive. In the first part of this paper, we prove that if $[D \to_c D]$ or $[D \to_s D]$ is bounded complete, then $D$ is distributive, which confirms that his conjecture is true.

Amadio (1991) [3] and Curien (1998) [4] raised the question of whether the category of stable bifinite domains (**SB** for short) in sense of Amadio–Droste is the largest Cartesian closed full subcategory of the category of $\omega$-algebraic meet-cpos with CM functions ($\omega$-**SAM** for short). In the second part of this paper, we prove that for any $\omega$-algebraic meet-cpo $D$ and certain non-distributive finite poset $\widetilde{M}$, if $[D \to_c \widetilde{M}]$, $[[D \to_c \widetilde{M}] \to_c [D \to_c \widetilde{M}]]$ and $[[[D \to_c \widetilde{M}] \to_c \widetilde{M}] \to_c [[D \to_c \widetilde{M}] \to_c \widetilde{M}]]$ are $\omega$-algebraic, then we have that (1) $D$ is finitary; (2) if $D$ is not stable bifinite, then $[[D \to_c \widetilde{M}] \to_c [D \to_c \widetilde{M}]]$ is not finitary. So, the category **SB** is a maximal Cartesian closed full subcategory of $\omega$-**SAM**, which gives a partial solution to the problem posed by Amadio and Curien.



## 1. Introduction

Let **CONT** and **ALG** be the categories of continuous domains and algebraic domains with Scott-continuous functions respectively. The Cartesian closedness of full subcategories of **CONT** and **ALG** plays an important role in theoretical computer science [1,12,15] and was investigated by Plotkin [14], Smyth [16], Jung [2] and so on. On one hand, Jung [1,2] systematically studied the Cartesian closed maximal full subcategories of **CONT** and **ALG**, proved that there exist exactly four Cartesian closed maximal full subcategories of **CONT** and **ALG**, respectively. On the other hand, motivated from the study of the sequential computation, Berry [5] introduced stable functions and the category of **dI**-domains which then were investigated by many authors [3,4,7–11,13,17–19]. Following the direction of Jung's study, Zhang [19] investigated the largest Cartesian closed category of stable domains, and obtained the following result.

**Theorem 1.1.** *[19] Let D be a Scott-domain and $[D \to_c D]$ its conditionally multiplicative (CM for short) functions space.*

*(i) If $[D \to_c D]$ has a countable basis, then D is finitary.*

*(ii) If D is finitary and $[D \to_c D]$ is bounded complete, then D is distributive.*

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As an immediate consequence of the above result, the category of **dI**-domains with CM functions is the largest Cartesian closed subcategory within $\omega$-algebraic bounded complete domains, with the exponential object being the stable function space.

Zhang [19] mentioned that if the finiteness condition is dropped, then Theorem 1.1 should still hold, but the proof may be complicated. In this paper we will show that his conjecture is true.

Amadio [3] and Droste [11] showed that the category of stable bifinite domains (**SB** for short) is Cartesian closed and the stable function space is the exponential object. Amadio [3,4] raised the question of whether the category **SB** of Amadio–Droste [3,11] is the largest Cartesian closed full subcategory of the category of $\omega$-algebraic meet-cpos with CM functions ($\omega$-**SAM** for short). Zhang and Jiang [20] showed that, for any $\omega$-algebraic meet-cpo $D$, if the CM function space $[D \to_c D]$ (with the Berry order) satisfies the so-called property **M**, then $D$ is finitary. In [21], Xi, Yang and Kou showed that for any $\omega$-algebraic meet-cpo domain $D$ and the diamond lattice $M$ (the classical nondistributive lattice), if $[D \to_c M]$, $[[D \to_c M] \to_c [D \to_c M]]$ and $[[[D \to_c M] \to_c M] \to_c [[D \to_c M] \to_c M]]$ are $\omega$-algebraic, then (1) $D$ is finitary; (2) if $D$ is not stable bifinite, then $[[D \to_c M] \to_c [D \to_c M]]$ is not finitary. So, the category **SB** is a maximal Cartesian closed full subcategory of $\omega$-**SAM**. By these two results, if a subcategory $\mathcal{C}$ containing the diamond lattice $M$ as an object (or containing an object $D$ which has a copy of $M$ as a retract) is a Cartesian closed subcategory of $\omega$-**SAM**, then $\mathcal{C}$ must be a subcategory of **SB**. The diamond lattice $M$ plays an important role in the proof. But an algebraic meet-cpo may not contain a copy of $M$. In this paper, we show that $M$ can be replaced by certain finite posets, which gives a partial solution to the related question.

The rest of the paper is structured along the lines of preliminaries, distributivity, property finitary, and maximality of **SB**.

## 2. Preliminaries

We assume readers are familiar with the background on algebraic cpos. We here recall some basic definitions about CM functions and the Berry order which are due to Berry [5].

**Definition 2.1.** Let $D$ be a cpo (with bottom). For any $x, y \in D$, $\{x, y\}$ is called *compatible*, denoted by $x \uparrow y$, if $\{x, y\}$ is bounded above, i.e., there exists $z \in D$ such that $x, y \leq z$. $D$ is called a *meet-cpo* if

(a) for any $x, y \in D$, $x \wedge y$ exists whenever $\{x, y\}$ is compatible,

(b) if $R \subseteq D$ is a directed set and $x$ is compatible with the join of $R$, then

$$x \wedge (\bigvee R) = \bigvee \{x \wedge r \mid r \in R\}.$$

Let $D$ and $E$ be algebraic meet-cpos. A function $f$ from $D$ to $E$ is called a *conditionally multiplicative* (CM for short) function if it is continuous and preserves meets of pairs of compatible elements, i.e.,

$$\forall x, y \in D, x \uparrow y \Rightarrow f(x \wedge y) = f(x) \wedge f(y).$$

Note that a CM function is also called a stable function in [19].

Let $[D \to_c E]$ be the set of CM functions from $D$ to $E$. Let $f, g \in [D \to_c E]$. Define

$$f \leq_c g \text{ iff } \forall x, y \in D, x \leq y \Rightarrow f(x) = f(y) \wedge g(x),$$

and $\leq_c$ is called the *Berry order*, also named the stable order in [19,20].

**Theorem 2.2.** *[20] Let $D$ and $E$ be meet-cpos and $f, g$ compatible CM functions in $[D \to_c E]$. Then*

$$a \uparrow b \Rightarrow f(a) \wedge g(b) = f(b) \wedge g(a).$$

For a directed set of CM functions, the least upper bound exists and it is determined by the pointwise ordering.

**Theorem 2.3.** *[20] Let $D$ and $E$ be algebraic meet-cpos. Let $\{f_i \mid i \in I\} \subseteq [D \to_c E]$ be directed with respect to the Berry order. Then $(\bigvee_{i \in I} f_i)(x) = \bigvee_{i \in I} f_i(x)$ for any $x \in D$.*

For compact elements $d, e$ of algebraic meet-cpos $D$ and $E$ respectively, we often make use of the *step function* $d \searrow e : D \to E$, which is defined by

$$(d \searrow e)(x) = \begin{cases} e, & x \geq d, \\ \perp, & \text{otherwise}. \end{cases}$$

**Definition 2.4.** [1] A cpo $D$ is called an *L-domain* if each principle ideal, as a subposet, is a complete lattice.