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Dynamic monopolies for degree proportional thresholds in connected graphs of girth at least five and trees

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ABSTRACT

Let G be a graph, and let $\rho \in [0,1]$. For a set D of vertices of G, let the set $H_\rho(D)$ arise by starting with the set D, and iteratively adding further vertices u to the current set if they have at least $\lceil \rho d_G(u) \rceil$ neighbors in it. If $H_\rho(D)$ contains all vertices of G, then D is known as an irreversible dynamic monopoly or a perfect target set associated with the threshold function $u \mapsto \lceil \rho d_G(u) \rceil$. Let $h_\rho(G)$ be the minimum cardinality of such an irreversible dynamic monopoly.

For a connected graph G of maximum degree at least $\frac{1}{\rho}$, Chang showed $h_\rho(G) \leq 5.83\rho n(G)$, which was improved by Chang and Lyuu to $h_\rho(G) \leq 4.92\rho n(G)$. We show that for every $\epsilon > 0$, there is some $\rho(\epsilon) \in (0,1)$ such that $h_\rho(G) \leq (2+\epsilon)\rho n(G)$ for every ρ in $(0, \rho(\epsilon))$, and every connected graph G that has maximum degree at least $\frac{1}{\rho}$ and girth at least 5. Furthermore, we show that $h_\rho(T) \leq \rho n(T)$ for every ρ in $(0,1]$, and every tree T that has order at least $\frac{1}{\rho}$.

1. Introduction

We consider finite, simple, and undirected graphs, and use standard terminology and notation.

Let G be a graph with vertex set $V(G)$. Let $\phi : V(G) \to \mathbb{N}_0$ be a threshold function such that $\phi(u)$ is at most the degree $d_G(u)$ of u in G for every vertex u of G. For a set D of vertices of G, let $H_{(G,\phi)}(D)$ be the smallest set $\bar{D}$ of vertices of G such that $D \subseteq \bar{D}$, and every vertex u in $V(G) \setminus \bar{D}$ has less than $\phi(u)$ neighbors in $\bar{D}$. Note that the set $H_{(G,\phi)}(D)$ can be constructed by starting with the set D, and iteratively adding further vertices u to the current set if they have at least $\phi(u)$ neighbors in it. Such iterative expansion processes have been considered in a variety of contexts [2,9–11,8,14,15,17,20,23]. If $H_{(G,\phi)}(D) = V(G)$, then D is a *ϕ-dynamic monopoly* of G. Let $h_\phi(G)$ be the minimum cardinality of a ϕ-dynamic monopoly of G.

By a simple probabilistic argument, very similar to the one used by Alon and Spencer [4] to prove the Caro–Wei bound on the independence number of a graph [8,22], Ackerman, Ben-Zwi, Wolfovitz [1] showed

$$h_\phi(G) \leq \sum_{u \in V(G)} \frac{\phi(u)}{d_G(u)+1}. \tag{1}$$

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Essentially the same bound was obtained independently by Reichman [21]. For an application of this argument to independence in hypergraphs see [5].

It is easy to see [11] that (minimum) dynamic monopolies and (maximum) generalized degenerate induced subgraphs are dual notions, that is, (1) can be considered the dual counterpart of bounds as in [3].

A very natural choice for the threshold function is to assign values that are proportional to the vertex degrees. Specifically, for some real parameter ρ in $[0, 1]$, let

$$\phi_\rho : V(G) \to \mathbb{N}_0 : u \mapsto \lceil \rho d_G(u) \rceil.$$

If D is a random set of vertices of G that contains each vertex independently at random with probability ρ, then, for every vertex u of G, the expected number of neighbors of u that belong to D is $\rho d_G(u)$. This suggests that $h_{\phi_\rho}(G)$ might be only slightly bigger than $\rho n(G)$, where $n(G)$ is the order of G. Without any restriction on ρ or G though, this intuition is misleading. In fact, if ρ is positive but much smaller than $\frac{1}{n(G)}$, then $h_{\phi_\rho}(G)$ is at least 1, while $\rho n(G)$ can be arbitrarily small. As observed by Chang [12], it is reasonable to consider only values of ρ that are at least $\frac{1}{\Delta(G)}$, where $\Delta(G)$ is the maximum degree of G, because $\phi_{\frac{1}{\Delta(G)}} = \phi_\rho$ for every ρ in $\left(0, \frac{1}{\Delta(G)}\right]$. For a connected graph G and $\rho \in \left[\frac{1}{\Delta(G)}, 1\right]$, Chang [12] proved

$$h_{\phi_\rho}(G) \le (2\sqrt{2} + 3)\rho n(G) \approx 5.83 \rho n(G), \tag{2}$$

which was improved by Chang and Lyuu [13] to

$$h_{\phi_\rho}(G) \le 4.92 \rho n(G). \tag{3}$$

Note that the bound in (1) might evaluate to $\Omega(n(G))$, because, for instance, vertices of degree 1 contribute $\frac{1}{2}$ rather than $O(\rho)$ to the right hand side of (1). In fact, especially for small values of ρ, and graphs with many vertices of small degrees, the bound (3) can be much better than the bound (1).

The proof strategies for (2) and (3) are quite different. The bound (2) is proved by a suitable adaptation of the argument of Ackerman, Ben-Zwi, Wolfovitz [1]. Vertices of small degree, that is, at most $\frac{1}{\rho}$, are treated differently from those of large degree, that is, more than $\frac{1}{\rho}$. A small set X_0 of vertices of large degree ensures that the remaining vertices of large degree have few neighbors of small degree outside of $H_{(G,\phi_\rho)}(X_0)$. This allows to apply the argument of Ackerman et al. to the vertices of large degree outside of X_0. The bound (3) is proved by a random procedure that considers a sequence $X_1, X_2, \ldots$ of random sets of vertices each containing every individual vertex independently at random with probability 3.51ρ. Starting with the empty set, a ϕ_ρ-dynamic monopoly is constructed by iteratively adding the vertices in $X_i \setminus H_{(G,\phi_\rho)}(X_1 \cup \ldots \cup X_{i-1})$ to the current set. Chernoff's inequality is used to ensure that $H_{(G,\phi_\rho)}(X_1 \cup \ldots \cup X_i)$ grows sufficiently fast. The proof of (3) has some resemblance to iterative random procedures that are used to show lower bounds on the independence number [16,18].

It is natural to ask for the best-possible constant in bounds of the form (2) and (3). We contribute to this question by showing the following results.

Theorem 1. *For every positive ϵ, there is some $\rho(\epsilon)$ in $(0, 1)$ such that*

$$h_{\phi_\rho}(G) \le (2 + \epsilon)\rho n(G) \tag{4}$$

for every ρ in $(0, \rho(\epsilon))$, and every connected graph G that has maximum degree at least $\frac{1}{\rho}$ and girth at least 5.

The proof of Theorem 1 is based on a combination of the techniques from [12,13]. Note that (4) requires a sufficiently small value of ρ, but that bounds like (2), (3), and (4) are especially interesting for small values of ρ. It is possible to generalize (4) to strongly connected directed graphs similarly as in [13].

Theorem 2. *If ρ is in $(0, 1]$, and T is a tree of order at least $\frac{1}{\rho}$, then*

$$h_{\phi_\rho}(T) \le \rho n(T). \tag{5}$$

Note that $h_{\phi_\rho}(T)$ can be computed in linear time [9] for a given tree.

In view of the upper bounds given by Theorems 1 and 2, it is natural to consider lower bounds as well. Unfortunately, "$h_{\phi_\rho}(G) \ge 1$" is the only lower bound that is valid for positive ρ and all connected graphs G, regardless of their girth and maximum degree, that is, stronger lower bounds require additional restrictions.

For many more references to and discussion of related work see [12,13].

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