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On the geometry and algebra of networks with state

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ABSTRACT

In [1] an algebra of automata with interfaces, **Span**(**Graph**), was introduced with main operation being communicating-parallel composition – a system is represented by an expression in this algebra. A system so described has two aspects: an informal network geometry arising from the algebraic expression, and a space of states and transition given by its evaluation in **Span**(**Graph**). Note that **Span**(**Graph**) yields purely compositional descriptions of systems.

In this paper we give an alternative globally (non-compositional) view of the same systems. In order to do this we make mathematically precise the network geometry in terms of monoidal graphs, and assignment of state in terms of morphisms of monoidal graphs. We call such globally described systems *networks with state*. To relate networks with state and **Span**(**Graph**) systems we use the algebra of cospans of monoidal graphs.

As an illustration we give a new representation of a (non-compositionally described) class of Petri nets in the compositional setting of **Span**(**Graph**). We also present a tool for calculating with networks with state.

Both algebras, of spans and of cospans, are symmetric monoidal categories with commutative separable algebra structures on the objects.

We include a short section, following [2], in which we show that a simple modification of the algebra allows the description of networks in which the tangling of connectors may be represented, yielding a connection with the theory of knots.

1. Introduction

In 1997 Katis, Sabadini and Walters introduced a parallel algebra of automata, **Span**(**Graph**), in [1] and at the same conference [3] showed how P/T nets could be embedded in this algebra. In 2000 [4] the same authors introduced a related sequential algebra of automata, namely **Cospan**(**Graph**) and showed how the two algebras combined could model hierarchical systems with evolving geometry. Many of the remarks we make in this paper apply both to **Span**(**Graph**) and **Cospan**(**Graph**) though we concentrate in this paper on the parallel algebra.

A system described in **Span**(**Graph**) (which we will call a **Span**(**Graph**) system) has two aspects: (i) an algebraic expression and (ii) its evaluation in **Span**(**Graph**). In [1] we showed how an algebraic expression yields an informal network geometry, and its evaluation an automaton of states and transitions of the system.

It is intrinsic to **Span**(**Graph**) systems that their geometry and state space are given compositionally. The aim of this paper is to give an alternative description of **Span**(**Graph**) systems, both of their network geometry and of their state space, a description which is global and non-compositional.

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Remark 1. All the systems we will consider are described at level of their control structure. Hence will be assumed finite, that is have a finite set of states and transitions. Moreover they will have only a finite number of components.

1.1. *The global geometry of* **Span**(**Graph**) *systems*

The algebra **Span**(**Graph**) is a symmetric monoidal category with extra structure and the informal pictorial representations of expressions in the algebra given in [1] followed the string diagrams [5] for expressions in monoidal categories. The diagrams used were not however formally justified, and shared the defect of string diagrams of being *progressive*, that is that composition is done from left to right (see Caveat 3.2 of [5]). This means that the diagrams are very close to the algebraic expressions they represent. The natural geometry of systems does not have this simple form – consider, for example, the geometry of a Petri net or of a circuit diagram. We give here instead a precise mathematical formulation of the global (non-compositional) geometry of **Span**(**Graph**) systems in terms of monoidal graphs which we believe correspond very well to natural system geometry.

The one possible objection to monoidal graphs as network geometries is that components have two sides (instead of n sides); this arises from the fact that we need to relate the global geometry to the algebra **Span**(**Graph**) which has nullary, unary, and binary operations.

1.2. *The global state space of* **Span**(**Graph**) *systems*

The second contribution of this paper is to show how not only the geometry but also the state space of a **Span**(**Graph**) system may be given globally and non-compositionally. We define what we call *networks with state* to be a morphism of monoidal graph from the network geometry to the large monoidal graph of Spans of graphs. This amounts to the assignment of state and transitions to each component and each connector of the network in a consistent way. The global space of states and transitions is then given by a limit.

1.3. *Relating global with compositional*

In order to prove that systems described globally as networks with state are the same as compositional **Span**(**Graph**) systems we need of course to describe networks with state compositionally. This involves introducing open networks (without states) as certain cospans of monoidal graphs and their algebra, as well as open networks with state.

Once we have made the connection we are able to give, as an illustration, a new representation of a (non-compositionally defined) class of Petri nets, C/E nets, as **Span**(**Graph**) systems, a representation more faithful than that of [3] to the geometry of Petri nets.

The separation of the two aspects of **Span**(**Graph**) systems into their geometry and their state space also permits us to give an efficient tool for calculating random behaviours of **Span**(**Graph**) systems.

1.4. *Earlier work*

The geometry of monoidal categories began with Penrose [6], and Joyal and Street [7] and is surveyed in [5] (though that papers does not consider the structure here discussed). The two main algebras of this paper, Spans and Cospans, were introduced by J. Benabou [8]. The algebra of **Span**(**Graph**), symmetric monoidal categories in which each object has a commutative separable algebra structure compatible with the tensor product, what have been called elsewhere [9] WSCC-categories (WSCC=well-supported compact closed) has been studied in detail by Sabadini and Walters with collaborators Katis and Rosebrugh, especially in [1,10,4,11–13], beginning with the work on relations with Carboni [14] in 1987.

The work has numerous antecedents in computer science – we mention just S. Eilenberg [15], S.L. Bloom, Z. Esik [16], Gh. Stefanescu [17]. The algebra has connections with quantum field theory [18], and as we will describe in this paper with the theory of knots.

There are many other models of networks consisting of components and connectors (see the introduction to [19] and references therein, though it fails to mention [1] which is earlier than most of the other models and influenced some) but they lack the connections with algebra, geometry and physics.

Other authors, like Baez, Fong, and collaborators, used an approach similar to our in order to describe an algebra of networks applied to control theory and circuit diagrams (see for example [20,21] and [22]).

1.5. *Outline of the paper*

Since to read the present paper it is crucial to have a thorough understanding of how systems are modelled in [1] we begin in the Section 2 with an informal review of the **Span**(**Graph**) model as well as of the related sequential algebra **Cospan**(**Graph**) [4]. The elements of **Span**(**Graph**) are *spans of graphs*, and the main operation is communicating-parallel composition of automata; the elements of **Cospan**(**Graph**) are *cospans of graphs* and the main operation is sequential composition of automata.

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