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# Non-overlapping matrices

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ABSTRACT

Two matrices are said non-overlapping if one of them cannot be put on the other one in a way such that the corresponding entries coincide. We provide a set of non-overlapping binary matrices and a formula to enumerate it which involves the $k$-generalized Fibonacci numbers. Moreover, the generating function for the enumerating sequence is easily seen to be rational.



## 1. Introduction

A string $u$ over a finite alphabet $\Sigma$ is called *self non-overlapping* (or equivalently *unbordered* or *bifix-free*) if it does not contain proper prefixes which are also proper suffixes. In other words, a string $u \in \Sigma^*$ is unbordered if it cannot be factorized as $u = vu'v$ with $v \in \Sigma^+$ and $u' \in \Sigma^*$. Nielsen in [13] provided the set $X \subset \Sigma^n$ of all bifix-free strings by means of a recursive construction. More recently, several researches [4,7–9] have been conducted in order to define particular subsets of $X$ constituted by *non-overlapping* (or *cross-bifix-free*) strings: two $n$ length strings $u, v \in X$ are called non-overlapping if any non-empty proper prefix of $u$ is different from any non-empty proper suffix of $v$, and *vice versa*.

In [3] the notion of unbordered strings is generalized to the two dimensional case by means of *unbordered pictures* which are rectangular matrices over $\Sigma$ by imposing that all possible overlaps between two copies of the same picture are forbidden. In particular, the authors extend in two dimensions the construction of unbordered strings proposed in [13] and describe an algorithm to generate the set $U$ of all the unbordered pictures of fixed size $m \times n$.

The aim of the present paper is to find a subset of unbordered matrices which are non-overlapping. As well as the sets given in [4,7–9] are non-overlapping subsets (or cross-bifix-free subsets) of strings of $X$, in the same way the set we are going to present is a non-overlapping subset of matrices of $U$. Roughly speaking two unbordered matrices $A$ and $B$ are non-overlapping if all possible overlaps between $A$ and $B$ are forbidden. More precisely, we can imagine to make a rigid movement of $B$ on $A$ such that $B$ glides on $A$. At the end of each slipping, which can be geometrically interpreted as a translation in a given direction on the plane, a (non-empty) common area (in the sequel *control window*) is formed. This common area can be seen as the usual intersection between the two rectangular arrays containing the entries of $A$ and $B$, which is, in turn, a rectangular array constituted by a finite number of $1 \times 1$ cells of the discrete plane. Each cell of the control window contains an entry of $A$ and an entry of $B$. If in each cell of the window the entry of $A$ coincides with the entry of $B$, then such window is called *overlapping window* and $A$ and $B$ *overlapping matrices*. On the contrary, if for any translation we never find an overlapping window, $A$ and $B$ are called *non-overlapping matrices*. For example, the unbordered

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$$\begin{array}{ccccccc} 1 & 0 & 0 & 1 & 1 & & \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ & & 1 & 0 & 1 & 1 & 1 \end{array}$$

**Fig. 1.** An example of overlap.

matrices $A = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 1 & 0 \end{pmatrix}$ and $B = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \end{pmatrix}$ can be overlapped as in Fig. 1 where the control window is showed.

Actually, a first attempt in order to generalize the concept of non-overlap in two dimensions between two distinct matrices can be found in [5] where the authors define a set of *cross-bibifix-free* square matrices over a finite alphabet. For the sake of clearness, two square matrices are called cross-bibifix-free when, essentially, they are non-overlapping only along the direction of the main diagonal. Here, using a completely different approach, we consider translations in any direction on the plane and matrices which can be also rectangular matrices, even if they have only binary entries. In this way the definition of non-overlapping set of matrices we are going to propose seems to be very close to the natural generalization in two dimensions of the concept of non-overlapping set of strings.

As it often happens, the extension to the bidimensional case of a typical concept related to strings is carried on by taking into account matrices. There are several cases in the literature where this process is occurred. For example, in [10] a bidimensional variant of the string matching problem is considered for sets of matrices. Another interesting example is given by the extension of classical finite automata for strings to the two-dimensional rational automata for pictures introduced in [1]. Moreover, it is worth to mention the problem of the pattern avoidance in matrices [11], which is a typical topic in linear structures as permutations and words.

In Section 2 we formally define a set of binary matrices which are proved to be non-overlapping matrices. The cardinality of this set is given in Section 3 where we also show that it is related to the well-known $k$-generalized Fibonacci numbers.

## 2. A set of non-overlapping binary matrices

The definition of non-overlapping matrices given in the Introduction can be formalized in terms of blocks matrices. Indeed, the control window we have referred in the previous section is essentially a particular block whose dimensions impose the ones of the other blocks of the partition of the matrix.

**Definition 2.1.** *Let $\mathcal{M}_{m\times n}$ be the set of all the matrices with m rows and n columns. Two distinct matrices $A, B \in \mathcal{M}_{m\times n}$ are called* non-overlapping *if all the following conditions are satisfied by $A$ and $B$:*

- *there do not exist two block partitions*
$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \text{ and } B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$
*such that $A_{11}, B_{22} \in \mathcal{M}_{r\times s}$, with $1 \le r \le m-1$, $1 \le s \le n-1$, and either $A_{11} = B_{22}$, or $A_{12} = B_{21}$, or $A_{21} = B_{12}$, or $A_{22} = B_{11}$.*
- *there do not exist two block partitions*
$$A = \begin{bmatrix} A_{11} \\ A_{21} \end{bmatrix} \text{ and } B = \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix}$$
*such that $A_{11}, B_{21} \in \mathcal{M}_{r\times n}$, with $1 \le r \le m-1$, and either $A_{11} = B_{21}$, or $A_{21} = B_{11}$.*
- *there do not exist two block partitions*
$$A = \begin{bmatrix} A_{11} & A_{12} \end{bmatrix} \text{ and } B = \begin{bmatrix} B_{11} & B_{12} \end{bmatrix}$$
*such that $A_{11}, B_{12} \in \mathcal{M}_{m\times s}$, with $1 \le s \le n-1$, and either $A_{11} = B_{12}$, or $A_{12} = B_{11}$.*

In other words, two distinct matrices are non-overlapping if any control window is not an overlapping window. Therefore, we can also define a *self non-overlapping* (or *unbordered*) matrix $A \in \mathcal{M}_{m\times n}$ as a matrix such that there does not exist a translation of $A$ on itself such that we can find an overlapping window. Clearly, this last definition can be easily deduced from Definition 2.1 with $A = B$ and suitably adapting the block partitions.

**Definition 2.2.** *A set $\mathcal{S}_{m\times n} \subset \mathcal{M}_{m\times n}$ is called* non-overlapping *if each matrix of $\mathcal{S}_{m\times n}$ is self non-overlapping and for any two matrices $A, B \in \mathcal{S}_{m\times n}$ they are non-overlapping matrices.*

Fixed the dimension $m \times n$ of the matrices, we now define a possible non-overlapping set where the matrices have a particular structure involving some of the entries on the frame of the matrix.