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On the decidability and complexity of problems for restricted hierarchical hybrid systems

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ABSTRACT

We study variants of a recently introduced hybrid system model, called a Hierarchical Piecewise Constant Derivative (HPCD). These variants (loosely called Restricted HPCDs) form a class of natural models with similarities to many other well known hybrid system models in the literature such as Stopwatch Automata, Rectangular Automata and PCDs. We study the complexity of reachability and mortality problems for variants of RHPCDs and show a variety of results, depending upon the allowed powers. These models form a useful tool for the study of the complexity of such problems for hybrid systems, due to their connections with existing models.

We show that the reachability problem and the mortality problem are co-NP-hard for bounded 3-dimensional RHPCDs (3-RHPCDs). Reachability is shown to be in PSPACE, even for n-dimensional RHPCDs. We show that for an unbounded 3-RHPCD, the reachability and mortality problems become undecidable. For a nondeterministic variant of 2-RHPCDs, the reachability problem is shown to be PSPACE-complete.

1. Introduction

Hybrid automata are an important class of mathematical model allowing one to capture both discrete and continuous dynamics in the same framework. There is currently much interest in *hybrid systems*, since they can be used to model many practical real world systems in which we have a discrete controller acting in a continuous environment. Their analysis has a huge range of potential applications, such as aircraft traffic management systems, aircraft autopilots, automotive engine control [1], chemical plants [2] and automated traffic systems for example.

Hybrid systems are described by a state-space model given by the Cartesian product of a discrete and continuous set. The system evolves over time according to a set of defined rules until some condition is satisfied, at which point a discrete, non-continuous event occurs. Such an event can cause an update to certain variables and change the continuous dynamics of the continuous variables.

A fundamental question concerning hybrid systems is that of *reachability*: does there exist a trajectory starting from some initial state (or set of states) which evolves to reach a given final state (or set of states) in finite time (defined formally in Section 2)? Related questions, such as *convergence* (does there exist a state, or periodic set of states, towards which the system converges for any initial state) or *control problems* (given an input, can the system be controlled to avoid some 'bad' set of states?), are also important, see [3], for example. Unfortunately, many reachability problems are *undecidable*, even

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for very restricted hybrid systems [4–7]. The objective of studying the decidability boundary is twofold; to obtain the most expressive system for which reachability is decidable and to study the simplest system for which it is undecidable.

An important and intuitive model of hybrid system is that of a *Piecewise Constant Derivative* (PCD) system. In this model, we partition the continuous state space into a finite number of nonempty regions, each of which is assigned a constant derivative defining the dynamics of a point within that region (see Section 2 for full details). It was proven in [8] that reachability for PCD systems in two dimensions (2-PCD) is decidable, but for three dimensions (3-PCD), the problem becomes undecidable [4]. One of the important properties of a PCD, which leads to its reachability problem being decidable in dimension two, is that trajectories can never 'cross' each other since each region has a constant derivative assigned. It can be proven that the trajectories are either periodic, or else form an expanding or contracting spiral which can be proven using geometric arguments on the *edge-to-edge successor* function of a 2-PCD.

In [9], a related model, called a *Hierarchical Piecewise Constant Derivative* (HPCD) system, was introduced. An HPCD is a 2-dimensional hybrid automaton where the dynamics in each discrete location is given by a 2-PCD (formal details are given in Section 2). Certain edges in the HPCD are called (transition) guards and cause the HPCD to change location if ever the trajectory reaches such an edge. When transitioning between locations, an affine reset rule may be applied. If all regions of the underlying PCDs are bounded, then the HPCD is called bounded. This model can thus be seen as an extension of a 2-PCD.

A 1-dimensional *Piecewise Affine Map* (1-PAM) is a piecewise function which is applied to the 1-dimensional real line, such that the function within each interval of the real line is affine (see Section 2 for details). The reachability problem for 1-PAMs is stated as an open problem in [10–14], but it becomes undecidable in the 2-dimensional case with fewer than 800 intervals [10]. In [14], 1-PAMs are proven to be equivalent to a 2-dimensional system called a planar pseudo-billiard system, also known as a "strange billiards" model in bifurcation and chaos theory [15] (see '*simulations*' under Section 2 for the definitions of equivalence and simulation). Some decidable results are known under restricted cases. In [12], it is proven that reachability is decidable for 1-dimensional Onto PAMs, which is a model such that every interval in the PAM can be exactly mapped to another. In [13], it is shown that for 1-PAMs over the integers (where all coefficients, the initial point and the final point are integers), the reachability problem is PSPACE-complete, which implies that reachability for rational 1-PAMs is at least PSPACE-hard. If PAMs are replaced by polynomials, the decidability of the reachability problem is open for any dimension [9]. If PAMs are replaced by piecewise rational maps, the reachability problem is undecidable even for dimension one [14].

Reachability for bounded 1-PAMs was shown to be equivalent to that of reachability for bounded HPCDs with either: i) comparative guards, identity resets and elementary flows in Proposition 3.20 of [12] or else ii) affine resets, non-comparative guards and elementary flows in Lemma 3.4 of [12] (see Section 2 for definitions). The authors of [12] also study reachability problems for PCDs defined on 2-*dimensional manifolds*, which we do not consider here.

Related to the reachability problem is the *mortality problem*. The mortality problem is the problem of determining if the trajectories starting from *all* initial points/configurations eventually halt (defined formally in Section 2). The mortality problem has been studied in many different contexts [16,13,11,17,18] and has connections with program verification, especially in a discrete setting. Similar to the case of reachability, the mortality for 1-PAMs is also stated as an open problem in [11, 13], and undecidability also starts at dimension two, in both the integer case [13], and for the rational case [11]. Global convergence is also known to be undecidable in dimension two [11], although both problems are decidable in dimension one when the piecewise affine function is continuous. The author of [13] also shows Π_2^0-completeness for the integer case.

However, neither reachability nor mortality is a superclass of the other. For the mortality problem, we must prove that *all* initial points will eventually halt, or else the system can be called *immortal* (meaning that the system may diverge, become periodic or quasi-periodic for example). Mortality for 1-PAMs over the integers is known to be PSPACE-complete [13]. Whether the undecidability results in dimension two still hold for a fixed number of intervals is unknown, in both the rational and integer cases.

Similarly to [12], we also aim to study the following question: "What is the simplest class of hybrid systems for which reachability is intractable or undecidable?" To this end, we define the model of *Restricted HPCD* (RHPCD), which is a deterministic bounded HPCD with elementary flows (derivatives of all continuous variables come from $\{0, \pm 1\}$), identity resets and non-comparative guards and is thus a simpler form of HPCD. These restrictions on the resets, derivatives and guards seem natural ones to consider. For example, restricting to identity resets means the trajectory will not have discontinuities in the continuous component, which is similar to a PCD trajectory. Restricting the derivatives to elementary flows ($\{0, \pm 1\}$) has similarities to a *stopwatch automaton*, for which all derivatives are from $\{0, 1\}$. Restricting the guards to be non-comparative gives strong similarities to the guards of a *rectangular automaton* [19], as well as the diagonal-free clock constraints of an *updatable timed automaton* [20].

We prove that a bounded 1-PAM can also be simulated by an RHPCD with arbitrary constant flows or with *linear* resets. Together with the results in [12], the reachability problem for bounded HPCDs is thus shown to be equivalent to that of bounded 1-PAMs when the HPCD only has one of the following: comparative guards, linear resets (or affine resets) or arbitrary constant flows, see Table 1 for an overview.

We then consider an n-dimensional analogue of RHPCDs, which we denote n-RHPCDs. We show that reachability is decidable (and in PSPACE) for bounded n-RHPCDs and mortality is decidable for bounded 2-RHPCDs. We show a lower bound that reachability and mortality for bounded 3-RHPCDs is co-NP-hard.

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