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Integer complexity: Representing numbers of bounded defect

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ABSTRACT

Define $\|n\|$ to be the *complexity* of n, the smallest number of ones needed to write n using an arbitrary combination of addition and multiplication. John Selfridge showed that $\|n\| \geq 3\log_3 n$ for all n. Based on this, this author and Zelinsky defined [4] the "defect" of n, $\delta(n) := \|n\| - 3\log_3 n$, and this author showed that the set of all defects is a well-ordered subset of the real numbers [1]. This was accomplished by showing that for a fixed real number s, there is a finite set S of polynomials called "low-defect polynomials" such that for any n with $\delta(n) < s$, n has the form $f(3^{k_1}, \ldots, 3^{k_r})3^{k_{r+1}}$ for some $f \in S$. However, using the polynomials produced by this method, many extraneous n with $\delta(n) \geq s$ would also be represented. In this paper we show how to remedy this and modify S so as to represent precisely the n with $\delta(n) < s$ and remove anything extraneous. Since the same polynomial can represent both n with $\delta(n) < s$ and n with $\delta(n) \geq s$, this is not a matter of simply excising the appropriate polynomials, but requires "truncating" the polynomials to form new ones.

1. Introduction

The *complexity* of a natural number n is the least number of 1's needed to write it using any combination of addition and multiplication, with the order of the operations specified using parentheses grouped in any legal nesting. For instance, $n = 11$ has a complexity of 8, since it can be written using 8 ones as

$$11 = (1+1+1)(1+1+1) + 1 + 1,$$

but not with any fewer than 8. This notion was implicitly introduced in 1953 by Kurt Mahler and Jan Popken [13]; they actually considered an inverse function, the size of the largest number representable using k copies of the number 1. (More generally, they considered the same question for representations using k copies of a positive real number x.) Integer complexity was explicitly studied by John Selfridge, and was later popularized by Richard Guy [9,10]. Following J. Arias de Reyna [5] we will denote the complexity of n by $\|n\|$.

Integer complexity is approximately logarithmic; it satisfies the bounds

$$3\log_3 n = \frac{3}{\log 3}\log n \leq \|n\| \leq \frac{3}{\log 2}\log n, \qquad n > 1.$$

The lower bound can be deduced from the result of Mahler and Popken, and was explicitly proved by John Selfridge [9]. It is attained with equality for $n = 3^k$ for all $k \geq 1$. The upper bound can be obtained by writing n in binary and finding a representation using Horner's algorithm. It is not sharp, and the constant $\frac{3}{\log 2}$ can be improved for large n [17].

Based on the above, this author and Zelinsky defined the *defect* of n:

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Definition 1.1. The *defect* of n, denoted $\delta(n)$ is defined by

$$\delta(n) := \|n\| - 3\log_3 n.$$

The defect has proven to be a useful tool in the study of integer complexity. For instance, one outstanding question regarding integer complexity, raised by Guy [9], is that of the complexity of 3-smooth numbers; is $\|2^n 3^k\|$ always equal to $2n+3k$, whenever n and k are not both zero? This author and Zelinsky used the study of the defect to show in [4] that this holds true whenever $n \le 21$.

This was accomplished by means of a method for, given a real number s, determining restrictions on what natural numbers n could have $\delta(n) < s$. They defined:

Definition 1.2. For a real number $s \ge 0$, the set A_s is the set of all natural numbers with defect less than s.

The method worked by first choosing a "step size" $\alpha \in (0,1)$, and then recursively building up coverings for the sets $A_\alpha, A_{2\alpha}, A_{3\alpha}, \ldots$; obviously, any A_s can be reached this way. Using this method, one can show:

Theorem 1.3 *(Covering theorem). For any real $s \ge 0$, there exists a finite set $\mathcal{S}_s$ of multilinear polynomials such that for any natural number n satisfying $\delta(n) < s$, there is some $f \in \mathcal{S}_s$ and some nonnegative integers $k_1, \ldots, k_{r+1}$ such that $n = f(3^{k_1}, \ldots, 3^{k_r})3^{k_{r+1}}$. In other words, given s one can find $\mathcal{S}_s$ such that*

$$\{n : \delta(n) < s\} \subseteq \bigcup_{f \in \mathcal{S}_s} \{f(3^{k_1}, \ldots, 3^{k_r})3^{k_{r+1}} : k_1, \ldots, k_{r+1} \ge 0\}.$$

It actually proved more: in particular, the polynomials in Theorem 1.3 are not arbitrary multilinear polynomials, but are of a specific form, for which [1] introduced the term *low-defect polynomials*. In particular, low-defect polynomials are in fact *read-once polynomials*, as considered in [16] for instance. See Sections 2 and 3 for more on these polynomials.

This sort of theorem is more powerful than it may appear; for instance, one can use it to show that the defect has unusual order-theoretic properties [1]:

Theorem 1.4 *(Defect well-ordering theorem). The set $\{\delta(n) : n \in \mathbb{N}\}$, considered as a subset of the real numbers, is well-ordered and has order type ω^ω.*

But while this theorem gave a way of representing a covering of A_s, this covering could include extraneous numbers not actually in A_s. In this paper we remedy this deficiency, and show that the sets A_s themselves can be described by low-defect polynomials, rather than low-defect polynomials merely describing a covering for each A_s.

In order to establish this result, we introduce a way of "truncating" a low-defect polynomial f to a given defect s, though this replaces one polynomial f by a finite set of low-defect polynomials $\{g_1, \ldots, g_k\}$. If we truncate every polynomial in the set $\mathcal{S}_s$ to the defect s, we obtain a set $\mathcal{T}_s$ of low-defect polynomials so that for any natural number n, $\delta(n) < s$ if and only if $n = f(3^{k_1}, \ldots, 3^{k_r})3^{k_{r+1}}$ for some $f \in \mathcal{T}_s$ and some $k_1, \ldots, k_{r+1}$. So as stated above we are no longer merely covering the set A_s, but representing it exactly. Our main result is as follows.

Theorem 1.5 *(Representation theorem). For any real $s \ge 0$, there exists a finite set $\mathcal{T}_s$ of multilinear polynomials such that a natural number n satisfies $\delta(n) < s$ if and only if there is some $f \in \mathcal{T}_s$ and some nonnegative integers $k_1, \ldots, k_{r+1}$ such that $n = f(3^{k_1}, \ldots, 3^{k_r})3^{k_{r+1}}$. In other words, given s one can find $\mathcal{T}_s$ such that*

$$\{n : \delta(n) < s\} = \bigcup_{f \in \mathcal{T}_s} \{f(3^{k_1}, \ldots, 3^{k_r})3^{k_{r+1}} : k_1, \ldots, k_{r+1} \ge 0\}.$$

This theorem is a special case of a stronger result; see Theorem 4.9.

Note that it is possible that, for a given s, there will be more than one set $\mathcal{T}_s$ satisfying the conclusions of Theorem 1.5. In particular, it's not clear if the $\mathcal{T}_s$ generated by the methods of this paper will be minimal in size. We ask:

Question 1.6. *For a given s, what is the smallest size $g(s)$ of a set $\mathcal{T}_s$ as above?*

We can also ask what can be said about the function $g(s)$ as s varies. It is not monotonic in s; for instance, let us consider what happens as s approaches 1 from below. We use the classification of numbers of defect less than 1 from [4]. For any $s < 1$, there's a finite set of numbers m such that any n with $\delta(n) < 1$ can be written as $n = m3^k$ for some k. Or in other words, $\mathcal{T}_s$ necessarily consists of a finite set of constants. As s approaches 1 from below, the required number of these constants approaches infinity, i.e., $\lim_{s\to 1^-} g(s) = \infty$. However, $g(1)$ is certainly finite, since all but finitely many of the constants in the $\mathcal{S}_s$ for $s < 1$ can be grouped together into a single infinite family, 3-represented by the single low-defect

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